Physics 15c (Hoffman)
Lecture # 15
Tues, Oct 26, 2010

Electromagnetic Waves

Last time

* LC transmission line wave equation:
$$\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\Delta x}{L}\frac{\Delta x}{C}\frac{\partial^2 V(x,t)}{\partial x^2} \quad \text{(same for } q, I\text{)}$$
* impedance: $Z = \sqrt{L/C}$
* wave speed: $c_w = \sqrt{\frac{\Delta x}{L}\frac{\Delta x}{C}}$
* Ampere's Law: $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$
* Gauss' Law: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_0}$
* energy: $\langle P\rangle = \left\langle \frac{dE}{dt}\right\rangle = \frac{1}{2}V_0 I_0$
* momentum: $\langle F\rangle = \left\langle \frac{dp}{dt}\right\rangle = \frac{1}{2}\frac{V_0 I_0}{c_w}$

Where is the energy?

What does it mean that the energy is stored "in" L and C?

Energy in L is stored in the form of a magnetic field.
→ it's around, but not in the wire

$$\frac{dE_L}{dx} = \frac{1}{2}\frac{L}{\Delta x}I^2 = \frac{1}{2}\frac{\mu}{\pi}\ln\left(\frac{d}{a}\right)I^2$$

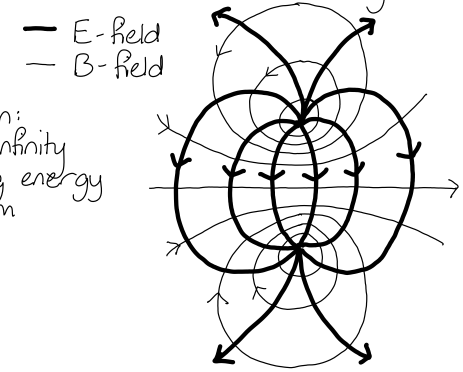
It depends on the μ of the material surrounding the wires.

Energy in C is stored in the form of an electric field.
→ it's between, but not in the wire

$$\frac{dE_C}{dx} = \frac{1}{2}\frac{C}{\Delta x}V^2 = \frac{1}{2}\frac{\pi\varepsilon}{\ln(d/a)}V^2$$

It depends on the ε of the material surrounding the wires.

The energy is not carried by the physical wires, but in the space around them, as an electromagnetic field.
→ wires are guiding the waves but not transmitting them



parallel wire transmission:
E & B fields extend to infinity
→ leaky way of carrying energy
→ radiation loss = problem

Poynting vector

$$\vec{S} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\vec{B}}{\mu}$$

* perpendicular to both $\vec{E}$ and $\vec{B}$
  ⇒ points in direction of wave transmission
* units $\vec{E}$ (V/m) × $\vec{H}$ (A/m)
  ⇒ Watts/m² = how much power is flowing per unit cross-sectional area

Where is the wave?

The wave can be regarded as a current or voltage wave within the wires, but it is also a wave of the $\vec{E}$ and $\vec{B}$ fields surrounding the wires. The wires guide the $\vec{E}$ & $\vec{B}$ waves.

So let's look directly at the wave equations for $\vec{E}$ & $\vec{B}$ in free space.

Goals for today:

* Maxwell's equations
* Wave equation for $\vec{E}$ and $\vec{B}$
* Plane waves
* Poynting vector → power & intensity
* Polarization

E & M Units

We are using MKS in this course.

CGS is more "elegant".
Electric (E) and magnetic (B) fields have the same dimensions.

MKS uses Volts, Amps, ohms, etc.
More familiar units for real world applications
In MKS, E/B has dimensions L/T (velocity)

Conversions

Gaussian → MKS
$4\pi \to 1/\varepsilon_0$
$B \to B/c$

TABLE B.1 CONVERSION FACTORS

(*Note*: Every 3 (except for exponents) is short for 2.99792458, the numerical value of the speed of light.)

	SI (mks)		Gaussian (cgs)
Length	meter (m)	10^2	centimeter
Mass	kilogram (kg)	10^3	gram
Time	second (s)	1	second
Force	newton (N)	10^5	dyne
Energy	joule (J)	10^7	erg
Power	watt (W)	10^7	erg/second
Charge	coulomb (C)	3×10^9	esu (statcoulomb)
Current	ampere (A)	3×10^9	esu/second (statampere)
Electric field	volt/meter	$\frac{1}{3} \times 10^{-4}$	statvolt/centimeter
Potential	volt (V)	$\frac{1}{300}$	statvolt
Displacement	coulomb/meter^2	$12\pi \times 10^5$	statcoulomb/centimeter^2
Resistance	ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	second/centimeter
Capacitance	farad (F)	9×10^{11}	centimeter
Magnetic field	tesla (T)	10^4	gauss
Magnetic flux	weber (Wb)	10^8	maxwell
H	ampere/meter	$4\pi \times 10^{-3}$	oersted
Inductance	henry (H)	$\frac{1}{9} \times 10^{-11}$	second^2/centimeter

Maxwell's Equations

Gauss' Law — No name

$$\boxed{\begin{array}{ll} ① \ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & ③ \ \vec{\nabla}\cdot\vec{B} = 0 \\ ② \ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & ④ \ \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} + \mu_0\vec{J} \end{array}}$$

Faraday's Law $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B$ — Maxwell's correction — Ampere's Law

and $\vec{\nabla}$ is the vector (Cartesian):

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

[Gaussian units:
① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$
② $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$
③ $\vec{\nabla}\cdot\vec{B} = 0$
④ $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$]

In **vacuum**, $\rho = 0$ and $\vec{J} = 0$, so:

$$\boxed{\begin{array}{l} ① \ \vec{\nabla}\cdot\vec{E} = 0 \\ ② \ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ ③ \ \vec{\nabla}\cdot\vec{B} = 0 \\ ④ \ \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

Note: our "medium" (vacuum) obeys these equations, which describe the relationship between electric and magnetic fields. We're now going to look for a wave equation from these, for either E or B. So we'll try to eliminate one of them, and look for something relating space derivatives to time derivatives.

Let's try to eliminate B first, to get everything in terms of E.

Take the curl of both sides of ②

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

time derivative commutes with space derivative

Take the time derivative of both sides of ④

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Combine these:

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{E})} = -\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

use identity: $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0\ \text{(by Maxwell ① in vacuum)}})-(\vec{\nabla}\cdot\vec{\nabla})\vec{E}$$

$\Rightarrow$ $\boxed{\nabla^2\vec{E}=\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t}}$ ← **vacuum wave eqn**

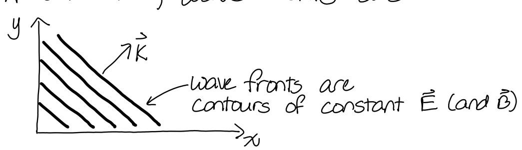
Can do the same thing with $\vec{B}$: Take $\vec{\nabla}\times$ ④, plug in time derivative of ②, apply BAC-CAB identity, arrive at:

$\boxed{\nabla^2\vec{B}=\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t}}$ ← **vacuum wave eqn**

These are 3-dim wave eqns (just like 2-dim wave eqns for rubber sheet)

Plane wave solutions

Plane waves travel in the $\vec{k}$ direction; wave fronts are planes perpendicular to $\vec{k}$.



$\boxed{\vec{E}=\vec{E}_0\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}}$ $\boxed{\vec{B}=\vec{B}_0\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}}$

Dispersion relation

$$\frac{\partial^2}{\partial t^2}\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)}=\frac{1}{\varepsilon_0\mu_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}_0\,e^{i(k_xx+k_yy+k_zz-\omega t)}$$

$$(i\omega)^2\vec{E}=\frac{1}{\varepsilon_0\mu_0}\left[(ik_x)^2+(ik_y)^2+(ik_z)^2\right]\vec{E} \quad\rightarrow\quad \omega^2=\frac{1}{\varepsilon_0\mu_0}|\vec{k}|^2$$

$\frac{\omega}{|\vec{k}|}=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=3\times10^8$ m/s $=c$ is the wave velocity $\Rightarrow$ $\boxed{\omega=c|\vec{k}|}$

Constraints

We found that $\vec{E}$ and $\vec{B}$ can be plane waves in any direction $\vec{k}$. But there are some additional constraints from the other Maxwell's equations.

① $\vec{\nabla}\cdot\vec{E}=0$

$$\frac{\partial}{\partial x}E_x+\frac{\partial}{\partial y}E_y+\frac{\partial}{\partial z}E_z=0$$

$$\frac{\partial}{\partial x}E_{0,x}e^{i(\vec{k}\cdot\vec{x}-\omega t)}+\frac{\partial}{\partial y}E_{0,y}e^{i(\vec{k}\cdot\vec{x}-\omega t)}+\frac{\partial}{\partial z}E_{0,z}e^{i(\vec{k}\cdot\vec{x}-\omega t)}=0$$

$$ik_xE_{0,x}e^{i(\vec{k}\cdot\vec{x}-\omega t)}+ik_yE_{0,y}e^{i(\vec{k}\cdot\vec{x}-\omega t)}+ik_zE_{0,z}e^{i(\vec{k}\cdot\vec{x}-\omega t)}=0$$

$\boxed{\vec{k}\cdot\vec{E}=0}$ $\rightarrow$ $\boxed{\begin{matrix}\vec{E}\perp\vec{k}\\ \vec{B}\perp\vec{k}\end{matrix}}$

③ $\vec{\nabla}\cdot\vec{B}=0$

→ same procedure finds $\boxed{\vec{k}\cdot\vec{B}=0}$

$\Rightarrow$ $\vec{E}$ and $\vec{B}$ are transverse waves (longitudinal $\vec{E}$ and $\vec{B}$ waves not allowed)

② $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

$$\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\left[\hat{x}(ik_x)+\hat{y}(ik_y)+\hat{z}(ik_z)\right]\times\vec{E}=i\omega\vec{B}$$

$\boxed{\vec{k}\times\vec{E}=\omega\vec{B}}$ $\Rightarrow$ $\vec{E}\perp\vec{B}$ and $|\vec{E}|=\frac{\omega}{|\vec{k}|}|\vec{B}|=c|\vec{B}|$

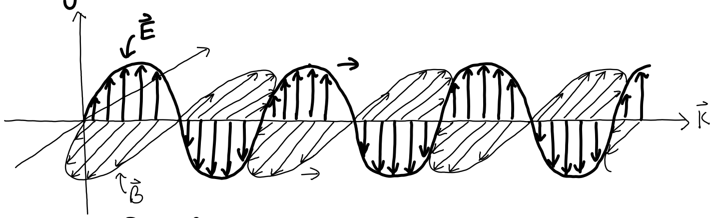
④ $\vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

→ same procedure finds $\boxed{\vec{k}\times\vec{B}=-\frac{\omega^2}{c}\vec{E}}$

So $\vec{E}$ and $\vec{B}$ are both perpendicular to $\vec{k}$, and perpendicular to each other. The relative magnitudes of $\vec{E}$ and $\vec{B}$ are also determined.

CONSTRAINTS
$\vec{E}\perp\vec{B}$
$\vec{E}\perp\vec{k}$
$\vec{B}\perp\vec{k}$
$|\vec{E}|=c|\vec{B}|$

Electromagnetic Wave



Pair of perpendicular, transverse waves travel along $\vec{k}$

Energy: Waves carry energy.

Poynting vector tells us how energy is carried.

$$\boxed{\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \frac{1}{\mu_0}\vec{B}}$$

Notice: $\vec{S}$ is parallel to $\vec{E}\times\vec{B}$, i.e. parallel to $\vec{k}$
→ energy is carried in the direction of $\vec{k}$, as expected.

Magnitude: $|\vec{S}| = |\vec{E}||\vec{H}| = |\vec{E}|\frac{1}{\mu_0}|\vec{B}| = |\vec{E}|\frac{1}{\mu_0 c}|\vec{E}|$

$$\boxed{|\vec{S}| = c\varepsilon_0 E^2}$$

$[E] = \frac{\text{force}}{\text{charge}} = \frac{\text{energy}}{\text{charge}\cdot\text{length}} = \frac{\text{voltage}}{\text{length}}$; $\frac{V}{m}$ in MKS

$[c\varepsilon_0] = \frac{m}{s}\cdot\frac{C^2}{Nm^2} = \frac{C/s}{(N/C)/m} = \frac{\text{current}}{\text{voltage}} = \frac{1}{\text{resistance}}$; $\frac{1}{\Omega}$ in MKS

Impedance

Vacuum impedance $= \boxed{Z_0 = \frac{1}{c\varepsilon_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega}$

Put all units together: $[c\varepsilon E^2] = \left(\frac{\text{volts}}{\text{meter}}\right)^2 \frac{1}{\text{ohm}} = \frac{V^2}{R}\frac{1}{L^2} = \frac{\text{power}}{\text{area}}$; $\frac{W}{m^2}$ in MKS

Time average of $|\vec{S}|$: $\left\langle \frac{E_0^2\cos^2(\vec{k}\cdot\vec{x}-\omega t)}{Z_0} \right\rangle = \boxed{\frac{1}{2}c\varepsilon_0 E_0^2 = \text{Intensity}}$

Note: ε_0, μ_0, c, and Z_0 are related to each other

$$\varepsilon_0 = \frac{1}{cZ_0} \qquad \mu_0 = \frac{Z_0}{c} \qquad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(→ only 2 of them are independent)

When we deal with EM waves in matter, we introduce

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} \Rightarrow c_w = \frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu}}, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}$$

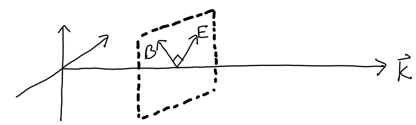
index of refraction; speed of wave; new impedance

Sometimes textbooks are sloppy, because $\mu \approx \mu_0$ almost always

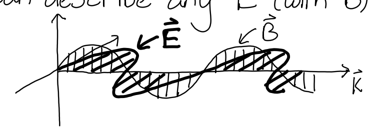
→ textbooks sometimes use $\frac{1}{n}$ and Z almost interchangeably in cgs units

Polarization

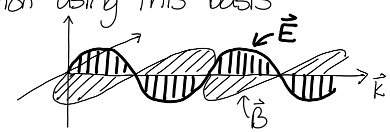
For a given $\vec{k}$ direction, $\vec{E}$ can point anywhere in plane $\perp$ to $\vec{k}$, with $\vec{B}\perp\vec{E}$



Can describe any $\vec{E}$ (with $\vec{B}$) direction using this basis



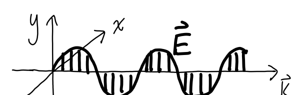
and



Linear combination of these gives all possible solutions for given $\vec{k}$.

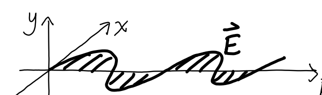
By convention, usually just think about $\vec{E}$-field (but keep in mind that $\vec{B}$ is there) and define the polarization by direction of $\vec{E}$.

So our basis is: (with $\vec{k} \parallel \hat{z}$)



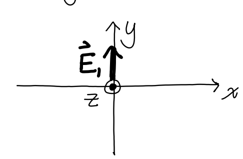
$\vec{E}_1 = E_0\hat{y}e^{i(kz-\omega t)}$

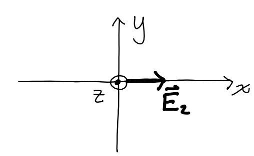
and



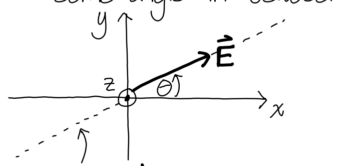
$\vec{E}_2 = E_0\hat{x}e^{i(kz-\omega t)}$

Or from the point of view looking down z-axis with EM wave coming at you...





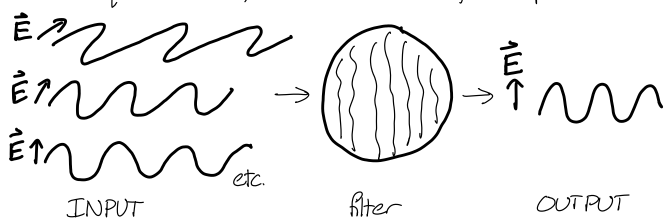
For some angle in between:



$$\vec{E} = \cos\theta\,\vec{E}_2 + \sin\theta\,\vec{E}_1$$
$$= E_0[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}]\,e^{i(kz-\omega t)}$$

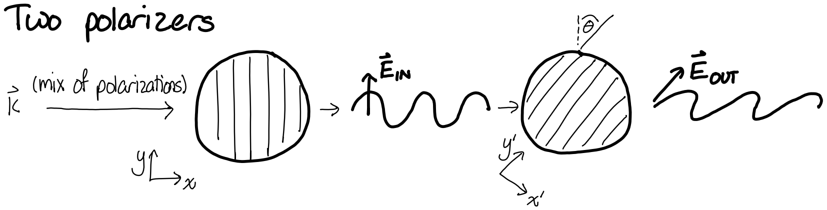
Called **linear polarization** as $\vec{E}$ stays along this line for all time and space as this wave propagates.

Most light sources have light with many EM waves, many polarizations...
Q: how do we make linearly polarized light?
A: with a polarizer
That is, a filter with long polymers
(invented by Edwin Land as a 19-yr-old Harvard ug
→ quit Harvard, founded Polaroid, shaped Science Center)



For all input polarizations, only the component of $\vec{E}$ lined up with polymers is transmitted.

Two polarizers



Light in between is vertically polarized

$$\vec{E}_{IN} = E_0\hat{y}\,e^{i(kz-\omega t)}$$
$$= E_0[\sin\theta\,\hat{x}' + \cos\theta\,\hat{y}']\,e^{i(kz-\omega t)} \quad \text{Rewrote in } \hat{x}', \hat{y}' \text{ coordinates}$$

$$\vec{E}_{OUT} = E_0[\cos\theta\,\hat{y}']\,e^{i(kz-\omega t)} \quad \text{Second polarizer transmits only component aligned with } \hat{y}'.$$

Notice that for $\theta = \frac{\pi}{2} = 90°$, $\vec{E}_{OUT} = 0$
⇒ crossed (⊥) polarizers extinguish all light

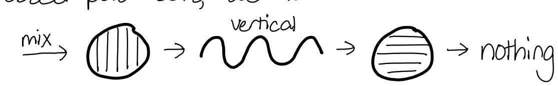
Malus' Law

① $|\vec{E}_{OUT}| = |\vec{E}_{IN}|\cos\theta$ where θ is the angle between input light polarization and the polarizer alignment

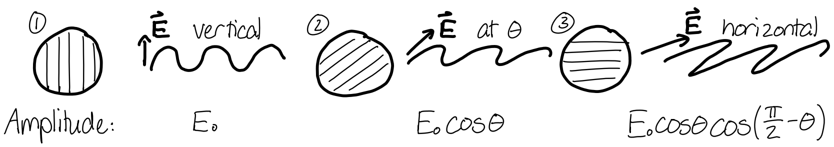
② Intensities:
$I_{OUT} = I_{IN}\cos^2\theta$ because recall, from a few pages ago, that $I = \frac{1}{2}c\varepsilon_0 E_0^2$, i.e. intensity scales like the square of E-field amplitude

Three polarizers

With two crossed polarizers, we had:



BUT



$$E_{OUT} = E_{IN}\cos\theta\sin\theta$$
$$= E_{IN}\cdot\frac{1}{2}\sin 2\theta$$

So $I_{OUT} = \left[\frac{1}{4}\sin^2(2\theta)\right] I_{IN}$

Quiz

(a) For what intermediate θ (of polarizer ②) is the power transmission maximized?

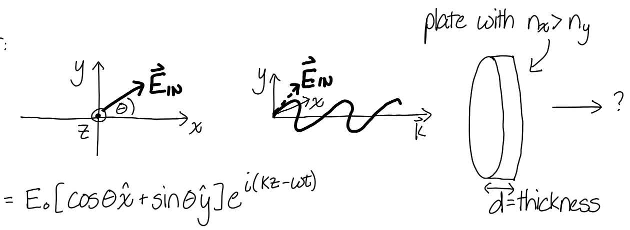
(b) What is the maximum transmitted power?

Circular Polarization

Speed of light depends on the surrounding medium.
$c_w = c/n$, where n is a material property

Consider a material where $\vec{E}_1$ and $\vec{E}_2$ basis vectors travel at different speeds, i.e. $n_x \neq n_y$
so the components of the EM wave will separate.

Input:



$\vec{E}_{IN} = E_0[\cos\theta\hat{x} + \sin\theta\hat{y}]e^{i(kz-\omega t)}$

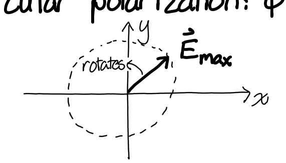
Qualitatively, the y-component travels faster, so when $\vec{E}_x$ and $\vec{E}_y$ emerge, they are out of phase with each other. Here, $\vec{E}_x$ will lag behind by a shift ϕ, where ϕ depends on n_x, n_y and d.

Suppose output is $\vec{E}_{OUT} = E_0[e^{-i\phi}\cos\theta\hat{x} + \sin\theta\hat{y}]e^{i(kz-\omega t)}$

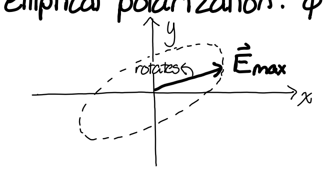
Real part $= E_0[\cos(kz-\omega t-\phi)\cos\theta\hat{x} + \cos(kz-\omega t)\sin\theta\hat{y}]$

So the y-component is maximized at a different time from the x-component. Direction of max $\vec{E}$ rotates!

circular polarization: $\phi = \pi/2$



elliptical polarization: $\phi \neq \pi/2$



Note: depending on relative values of n_x, n_y, and d, the maximum of $\vec{E}$ can rotate "clockwise" or "anti-clockwise".

Use your right thumb along the $\vec{k}$ axis of propagation, and your fingers curling around the rotation direction of $\vec{E}_{max}$, to name this as "right-handed" or "left-handed" circularly polarized light.

Summary

* Reviewed Maxwell's equations and derived wave equations for electric and magnetic fields
* Found that $\vec{E}$ and $\vec{B}$ are transverse waves and perpendicular to each other.
* Intensity (power/area) of EM waves: $I = \frac{1}{2}c\varepsilon_0 E_0^2$
* Polarization = direction of $\vec{E}$ for EM waves
  - → linear polarization
  - → circular polarization

Next time

* EM waves in matter
* reflection
* refraction

Quiz Answers

(a) 45°
(b) 25% of the intensity that made it through the first polarizer