Physics 15c (Hoffman)
Lecture #7
Thurs, Sept 23, 2010

Dispersion & Energy

Last time:

* continuous waves: $Ae^{i(kx-\omega t)}$ solves the wave eqn for <u>any</u> frequency ω
* dispersion relation: but k must take a particular value for each value of ω
  - → we find the relation between ω & k by plugging the solution $Ae^{i(kx-\omega t)}$ into the wave equation

  Note: the dispersion relation is a property of the <u>system</u> (i.e. the masses & springs themselves), not a property of the particular wave that moves through the system

* any function can be represented by a sum or integral of waves
  - sum: if a periodic function
  - integral: if a non-periodic function
  - → any function can be a wave as long as it propagates with the right velocity for the system in question
    - velocity: given by the dispersion relation
* "dispersion-less": linear dispersion relation
  - → all modes propagate w/ same phase velocity
  - → pulse maintains its shape
* "dispersion": non-linear dispersion relation
  - → modes propagate with different phase velocity
  - → pulse distorts over time
* phase velocity vs. group velocity

$$v_{phase} = \frac{\omega}{k} \qquad v_{group} = \frac{d\omega}{dk}$$

Goals for today:

* example of non-linear dispersion: plasma in ionosphere
* how does a pulse actually propagate in dispersive medium?

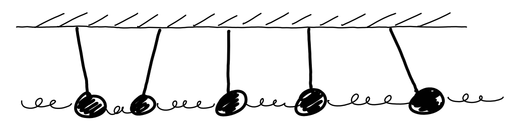
Dispersion relations:

We've studied 2 kinds of continuous waves:

1. dispersionless

$$\omega = \sqrt{\frac{E}{\rho\ell}}\,k$$

2. dispersive



$$\omega^2 = \frac{E}{\rho\ell}k^2 + \omega_0^2$$

What's the physical difference?

1. only has local interactions
   if you took away the interactions (remove the springs) then the components of the system could not oscillate by themselves

2. has local interactions plus a resonance
   if you took away the interactions (remove the springs) then each component would <u>still</u> oscillate independently with resonant frequency $\omega_0 = \sqrt{g/\ell}$

Many interesting systems have a resonance in addition to local interactions.

Plasma in the Earth's Ionosphere

UV radiation ionizes atoms in the upper atmosphere:

too high in the atmosphere → low density gas, so rarefied plasma (we have the resonance but no connecting springs → no propagation)

too low in the atmosphere → UV flux attenuated (little ionization)

"sweet spot" = ionosphere = altitudes of ~200-400 km

The resulting plasma is (overall) electrically neutral, but there can be displacements of charge density.

derivation of wave equation: (see H&L p168)

① $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{d\vec{E}}{dt} + \mu_0\vec{J}$

② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

} Maxwell's eqns

③ $m\frac{d^2\vec{x}}{dt^2} = -e\vec{E}$ } equation of motion for an electron

④ $\vec{J} = -n_0 e\frac{d\vec{x}}{dt}$ } current is just motion of electrons (n_0 is density of electrons)

Combine ③ and ④: $\frac{\partial\vec{J}}{\partial t} = \frac{n_0 e^2}{m}\vec{E}$ plug in

Time derivative of ①: $\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} + \mu_0\frac{\partial\vec{J}}{\partial t}$

Space derivative of ②: $\nabla^2\vec{E} = \vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$

Combine: $\nabla^2\vec{E} = \underbrace{\mu_0\varepsilon_0}_{\frac{1}{c^2}}\left(\frac{\partial^2\vec{E}}{\partial t^2} + \underbrace{\frac{n_0 e^2}{m\varepsilon_0}}_{\omega_p^2 = \text{resonant frequency of plasma}}\vec{E}\right)$

Let's keep it simple, look just at 1-dim for now:

$$\boxed{\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 E}{\partial t^2} + \omega_p^2 E\right)}$$ ← wave equation in a plasma

Note: this equation is a property of the plasma itself, which describes the motion of any wave that tries to propagate through that plasma.

derivation of dispersion relation from wave equation

Plug in our canonical solution: $Ae^{i(kx-\omega t)}$
↳ (into the wave eqn we just derived)

$$-k^2Ae^{i(kx-\omega t)} = \frac{1}{c^2}\left(-\omega^2Ae^{i(kx-\omega t)} + \omega_p^2Ae^{i(kx-\omega t)}\right)$$

$$\Rightarrow \boxed{\omega^2 = c^2k^2 + \omega_p^2}$$ ← dispersion relation for a plasma

compute phase & group velocities from dispersion relation

$\omega(k) = \sqrt{c^2k^2 + \omega_p^2}$

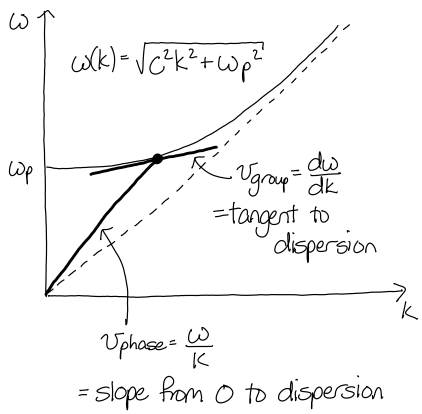
$v_{phase} = \frac{\omega}{k} = \frac{\sqrt{c^2k^2+\omega_p^2}}{k}$ $\qquad v_{group} = \frac{d\omega}{dk} = \frac{c^2k}{\sqrt{c^2k^2+\omega_p^2}}$

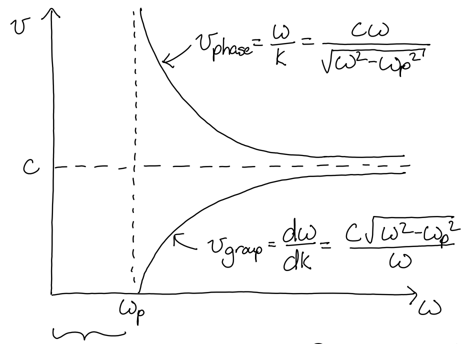
Often, it's more convenient to express these in terms of ω:

$k(\omega) = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$

$v_{phase} = \frac{c\omega}{\sqrt{\omega^2-\omega_p^2}}$ $\qquad v_{group} = c\,\frac{\sqrt{\omega^2-\omega_p^2}}{\omega}$

plot dispersion, v_{group}, & v_{phase}





no propagating solutions for $\omega < \omega_p$!

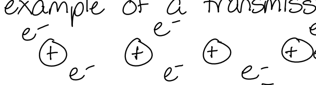
Physically, what is going on in our plasma?

$\omega_p = \sqrt{\frac{n_0 e^2}{m\varepsilon_0}}$ = frequency of oscillation if you displace a few electrons (they will be restored by the local + charge left behind)

For typical free electron densities ($N \sim 10^4 - 10^6\ e^-/cm^3$) in the ionosphere, this corresponds to frequencies ~10-30 MHz

Electron density n_0 tends to fall at night (no UV to ionize)
⇒ ω_p also ↓ after dark

This is another example of a transmissive system with an intrinsic resonance:



heavy ions (e.g. proton is 2000x heavier than e⁻)
⇒ these don't move much

once ionized, these are free to move

ω_p is a result of e^- "talking to" ⊕
→ restoring force btwn stationary ions & mobile e^-

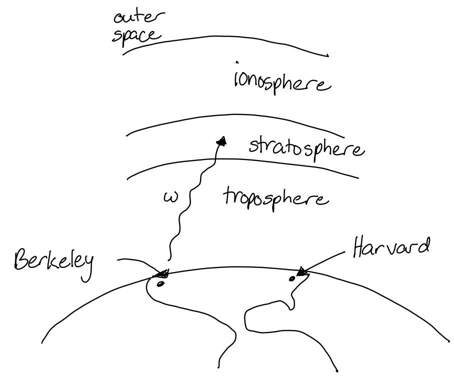
c is a result of e^- "talking to" e^-
→ they will repel each other

Why is this interesting?

1. $\omega > \omega_p$ → traveling waves described by dispersion relation
2. $\omega = \omega_p$ → uniform (coherent) oscillation over entire space
3. $\omega < \omega_p$ → exponentially attenuating with distance
   BUT the energy of the wave has to go somewhere, so it is (at least partially) reflected

How does this relate to EM waves (radio waves) in the ionosphere?

Suppose we send an AM radio wave up with $\omega < \omega_p$



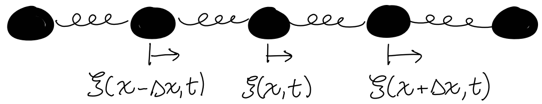
wave cannot propagate through ionosphere if $\omega < \omega_p$
→ will be reflected
→ basis of long-distance radio communication

At night, ω_p falls, because there is less ionization, so N falls.

Europe broadcasting:

5pm - 5am : 6.2 MHz
5am - 7am : 9.4 MHz
7am - 5pm : 15.5 MHz

Energy transmission of a wave



Consider a right-moving wave: $\xi(x,t) = \xi_0 \cos(kx-\omega t)$

Energy is in 2 forms:

① kinetic energy of the masses

② potential energy of the springs

Kinetic energy

velocity of the mass at x is $v(x,t) = \frac{\partial \xi(x,t)}{\partial t}$

$$\Rightarrow E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{\partial \xi(x,t)}{\partial t}\right)^2$$

there is a mass at every Δx

→ compute the linear energy density in the continuum limit (J/m)

$$\frac{dE_K}{dx} = \lim_{\Delta x \to 0} \frac{E_K}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{2}\frac{m}{\Delta x}\left(\frac{\partial \xi(x,t)}{\partial t}\right)^2 = \frac{1}{2}\rho_e\left(\frac{\partial \xi(x,t)}{\partial t}\right)^2$$

linear mass density

Potential energy

spring between x and $x+\Delta x$ has an energy of

$$E_S = \frac{1}{2}k_s\left(\xi(x+\Delta x,t) - \xi(x,t)\right)^2$$

Taylor expand: $\xi(x+\Delta x,t) = \xi(x,t) + \Delta x \frac{\partial \xi(x,t)}{\partial x}$

$$= \frac{1}{2}k_s\left(\frac{\partial \xi(x,t)}{\partial x}\Delta x\right)^2$$

there is a spring at every Δx

→ compute the linear energy density in the continuum limit (J/m)

$$\frac{dE_S}{dx} = \lim_{\Delta x\to 0}\frac{E_S}{\Delta x} = \lim_{\Delta x \to 0}\frac{1}{2}\frac{k_S}{\Delta x}\left(\frac{\partial \xi(x,t)}{\partial x}\Delta x\right)^2$$

$$= \lim_{\Delta x\to 0}\frac{1}{2}k_S\Delta x\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2 = \frac{1}{2}E\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2$$

elastic constant

In the continuum limit, we have:

$$\boxed{\frac{dE_K}{dx} = \frac{1}{2}\rho_e\left(\frac{\partial \xi(x,t)}{\partial t}\right)^2} \qquad \boxed{\frac{dE_S}{dx} = \frac{1}{2}E\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2}$$

A solution to the wave equation must take the specific form:

$\xi(x,t) = f(x \pm c_w t)$

→ plug this in to the expressions for energy density

$$\frac{\partial \xi(x,t)}{\partial t} = \frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial t} = \pm c_w f'(x \pm c_w t)$$

where $u = x \pm c_w t$

$$\frac{\partial \xi(x,t)}{\partial x} = \frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial x} = f'(x \pm c_w t)$$

$$\Rightarrow \frac{dE_K}{dx}(x,t) = \frac{1}{2}\rho_e c_w^2\left(f'(x\pm c_w t)\right)^2 = \frac{1}{2}E\left(f'(x \pm c_w t)\right)^2 = \frac{dE_S}{dx}(x,t)$$

where $c_w = \sqrt{\frac{E}{\rho_e}}$

⇒ At any given spatial location, at any given time,

$\boxed{\text{Kinetic energy density} = \text{Potential energy density}}$!!!

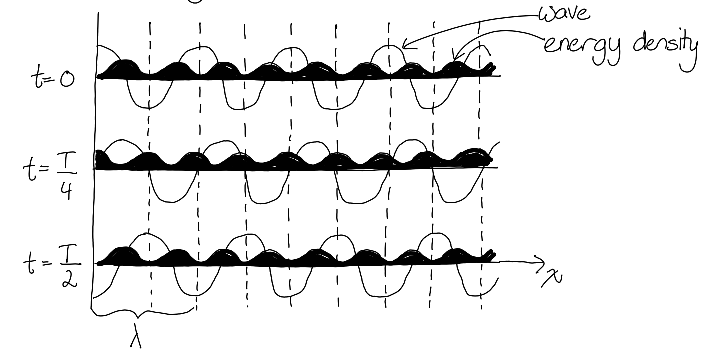
This may be counter-intuitive at first, because we are accustomed to thinking about single oscillators where the energy sloshes back & forth btwn kinetic & potential.

Note: we derived $\frac{dE_K}{dx}(x,t) = \frac{dE_S}{dx}(x,t)$ for any general wave $f(x \pm c_w t)$ but let's look at it in more detail for a specific non-dispersive wave $\xi_0\cos(kx-\omega t)$ [where $\omega = c_w k$]

$$\Rightarrow \frac{dE_{tot}}{dx}(x,t) = \frac{dE_K}{dx}(x,t) + \frac{dE_S}{dx}(x,t)$$

$$= \rho_e\left(\frac{\partial}{\partial t}\xi_0\cos(kx-\omega t)\right)^2 = \rho_e\omega^2\xi_0^2\sin^2(kx-\omega t)$$

Energy density comes in packets



- energy density comes in packets
- packets travel with the wave
- the velocity at which the wave (& therefore energy) moves is c_w

$\Rightarrow$ the energy flowing through a given point is

$$\frac{dE}{dt} = c_w \frac{dE}{dx}$$

Although energy density comes in packets, let's look at averages for a moment, to try to understand the big picture of where the energy is coming from & where it is going.

Energy density $\frac{dE}{dx} = \rho_e \omega^2 \xi_0^2 \sin^2(kx-\omega t)$

$\rightarrow$ averages to $\left\langle \frac{dE}{dx} \right\rangle = \frac{1}{2}\rho_e \omega^2 \xi_0^2 \Big\}$ average energy density of a wave

This all travels at $c_w = \sqrt{E/\rho_e}$ ($\leftarrow E$ = elastic constant here)

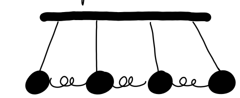
$\rightarrow$ the rate of energy transfer (in Joules/second = Watts) is:

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} c_w \rho_e \omega^2 \xi_0^2 = \frac{1}{2}\underbrace{\sqrt{E\rho_e}}_{\text{what is this?}}\,\omega^2 \xi_0^2 \Big\} \text{ average power of a wave}$$

$\sqrt{E\rho_e}$ is called the "impedance" of the medium (often called Z)

Note: impedance is a property of the medium, not of the wave itself.

Question: how does this work for dispersive media?
(Previous derivation was just for simple, non-dispersive mass-spring.)



$\omega = c_w k$

$\left(c_w = \sqrt{\frac{E}{\rho_e}} = v_{phase} \right)$

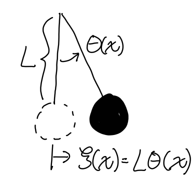
$$\frac{dE_{tot}}{dx} = \frac{dE_k}{dx} + \frac{dE_s}{dx} = 2\frac{dE_k}{dx} = 2\frac{dE_s}{dx}$$

$$= \rho_e \left(\frac{\partial \xi(x,t)}{\partial t} \right)^2$$

$\omega^2 = c_w^2 k^2 + \omega_0^2$

$\left(c_w = \sqrt{\frac{E}{\rho_e}}\ ;\ \omega_0 = \sqrt{\frac{g}{L}} \right)$

$$\frac{dE_{tot}}{dx} = \frac{dE_k}{dx} + \frac{dE_s}{dx} + \frac{dE_g}{dx}$$



$$E_g = mgL(1-\cos\theta)$$
$$= mgL\left(1-\left(1-\tfrac{1}{2}\theta^2\right)\right)$$
$$= mgL\left(\tfrac{1}{2}\theta^2\right)$$
$$= mgL\left(\frac{1}{2}\frac{\xi(x)^2}{L^2}\right)$$
$$= \frac{1}{2} m \frac{g}{L} \left(\xi(x)\right)^2$$

$$\Rightarrow \frac{dE_g}{dx} = \lim_{\Delta x \to 0} \frac{1}{2}\frac{m}{\Delta x}\frac{g}{L}\left(\xi(x)\right)^2 = \frac{1}{2}\rho_e \omega_0^2 \left(\xi(x,t)\right)^2$$

$$\frac{dE_{tot}}{dx} = \rho_e \left(\frac{\partial \xi(x,t)}{\partial t} \right)^2 + \frac{1}{2}\rho_e \omega_0^2 \left(\xi(x,t)\right)^2$$

Whatever the form of the energy density, we can still say:

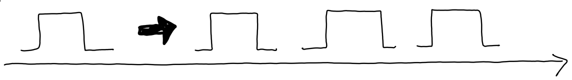
- For a wave at a single fixed frequency, the energy travels with the wave at the phase velocity. The energy flowing through a given point is therefore: $\frac{dE_{tot}}{dt} = v_{phase} \frac{dE_{tot}}{dx}$
- Complications arise when we consider a pulse containing more than one frequency, because we have to add individual frequency components linearly before squaring the total to get the energy.

How does a real pulse disperse?

And what is v_{group} really, anyway?

Imagine a pulse being sent over a distance:

non-dispersive



* on a non-dispersive medium, pulse shape is unchanged
* because all normal modes (all the sines & cosines that add together to give the pulse) have the same v_{phase}

dispersive

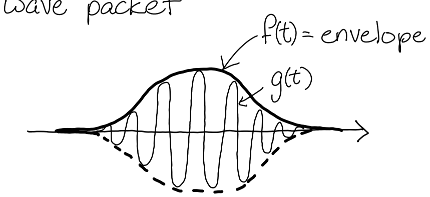
* on a dispersive medium, the pulse shape must change
* pulse shape gets "dispersed"

→ dispersion makes a poor medium for communication!

Wave packets

Information is carried by wave packets
→ consider a general wave packet

$$g(t) = \underbrace{f(t)}_{\text{pulse shape}} \underbrace{e^{-i\omega_c t}}_{\text{carrier wave}}$$



→ modulate the carrier wave $e^{-i\omega_c t}$ with a pulse $f(t)$

Now we're going to use Fourier analysis to understand how this packet propagates. This will lead naturally to a derivation of $v_{group} = \frac{d\omega}{dk}$ (recall: last time we sort of jusified it using just 2 waves & looking at the velocity of their beats)

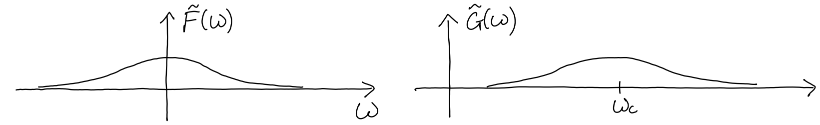
Fourier integral of the wave packet is:

$$\tilde{G}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega_c t}e^{i\omega t}dt$$

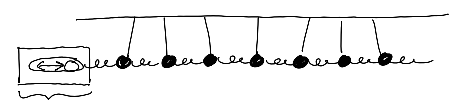
$$= \tilde{F}(\omega-\omega_c) \quad \text{where} \quad \tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

The frequency spread of the wave packet $\tilde{G}(\omega)$ has the same shape as the frequency spread of just the pulse envelope $\tilde{F}(\omega)$, but $\tilde{G}(\omega)$ is centered at ω_c, while $\tilde{F}(\omega)$ is centered around 0



check: $\tilde{G}(\omega_c) = \tilde{F}(\omega_c-\omega_c) = \tilde{F}(0)$ ✓

Now we look at how the whole wave packet $g(t)$ propagates in space:



motor drives
$g(t) = f(t)e^{-i\omega_c t}$

Forward-going wave packet is generated at $x=0$ as:

$$\xi(0,t) = g(t) = \underbrace{\int_{-\infty}^{\infty}\tilde{G}(\omega)e^{-i\omega t}d\omega}_{\text{inverse Fourier transform}}$$

remember: $\tilde{G}(\omega) \neq 0$ only near $\omega = \omega_c$
(say within $\Delta\omega$ of ω_c)

→ we know how each individual normal mode travels

$$e^{-i\omega t} \text{ at } x=0 \rightarrow e^{i(kx-\omega t)}$$

→ so the total wave packet should travel as the sum (integral)

$$\xi(x,t) = \int_{-\infty}^{\infty} \tilde{G}(\omega)\, e^{i(kx-\omega t)}\, d\omega$$

where this k is $k(\omega)$ from the dispersion relation

$$\xi(x,t) = \int_{-\infty}^{\infty} \tilde{G}(\omega) e^{i[k(\omega)x-\omega t]}\, d\omega$$

$$= \int_{\omega_c-\Delta\omega}^{\omega_c+\Delta\omega} \tilde{G}(\omega) e^{i[k(\omega)x-\omega t]}\, d\omega$$

(where $2\Delta\omega$ is the full width of the frequency range contained in this particular pulse.)

$$= \int_{\omega_c-\Delta\omega}^{\omega_c+\Delta\omega} \tilde{F}(\omega-\omega_c) e^{i[k(\omega)x-\omega t]}\, d\omega$$

Taylor expand: $k(\omega) = k(\omega_c) + \left.\frac{dk}{d\omega}\right|_{\omega_c}(\omega-\omega_c) + \frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}(\omega-\omega_c)^2 + \ldots$

change variables: $\omega' = \omega - \omega_c$

$$= \int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{i\left[\left(k_c + \omega'\left.\frac{dk}{d\omega}\right|_{\omega_c} + \frac{1}{2}(\omega')^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}\right)x - (\omega_c+\omega')t\right]}\, d\omega'$$

$$= e^{i(k_c x-\omega_c t)} \int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{i\omega'\left[\left.\frac{dk}{d\omega}\right|_{\omega_c}x - t\right]} \underbrace{e^{\frac{i}{2}(\omega')^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}x}}_{\text{This is the 2nd order Taylor term, presumably small, so let's ignore it for now}}\, d\omega'$$

$$\xi(x,t) = e^{i(k_c x-\omega_c t)} \underbrace{\int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{-i\omega'\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_c}x\right)}}\, d\omega'$$

This part looks like an inverse Fourier transform except that t is shifted by $\left.\frac{dk}{d\omega}\right|_{\omega_c}x$

$$\xi(x,t) = \underbrace{e^{i(k_c x-\omega_c t)}}_{\text{carrier wave}}\ \underbrace{f\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_c}x\right)}_{\text{shape of the wave packet}}$$

What did we expect to find?

For a non-dispersive medium, we just expect that the shape $f(t)$ propagates with some velocity v_{group}, so that at any time & position, it would look like $f(t - x/v_{group})$

Ah-hah! So we can identify $\frac{1}{v_{group}} = \left.\frac{dk}{d\omega}\right|_{\omega_c}$

$$\Rightarrow \boxed{v_{group} = \left.\frac{d\omega}{dk}\right|_{\omega_c}} \leftarrow \textbf{derivation of } v_{group}$$

Using only the first term of the Taylor expansion in the exponent, we approximated this as a dispersionless wave.

For a dispersive medium, things get more algebraically involved...

$$\xi(x,t) = e^{i(k_c x-\omega_c t)} \int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{i\omega'\left[\left.\frac{dk}{d\omega}\right|_{\omega_c}x - t\right]} \underbrace{e^{\frac{i}{2}(\omega')^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}x}}_{\text{keep this term too now}}\, d\omega'$$

But everything in this messy expression is, in principle, known. Presumably, you know the wave eqn and therefore the dispersion relation for your medium, so you can compute $\frac{dk}{d\omega}$ and $\frac{d^2k}{d\omega^2}$

And presumably you know the shape of the pulse you started with, $g(t)$, so you can ask Mathematica to calculate:

$$\tilde{F}(\omega') = \tilde{F}(\omega-\omega_c) = \tilde{G}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{i\omega t}\,dt$$

Then you can put all the pieces together and ask Mathematica to calculate:

$$\xi(x,t) = e^{i(k_c x-\omega_c t)}\int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega')\, e^{i\omega'\left[\frac{dk}{d\omega}\big|_{\omega_c} x - t\right]} e^{\frac{i}{2}(\omega')^2\frac{d^2k}{d\omega^2}\big|_{\omega_c} x}\, d\omega'$$

So taking a step back, this is pretty sweet:

We started knowing just the dispersion relation of our medium, $\omega(k)$, and the shape of our pulse $g(t)$ at the origin $x=0$.
By breaking it into a sum of normal modes (i.e. Fourier transforming it) we were able to:

① derive the group velocity $v_{group} = \frac{d\omega}{dk}$

② write an expression for $\xi(x,t)$ = the evolution of $g(t)$ across all space, for all times
i.e. we wrote down an analytic formula for how the pulse shape changes as it propagates

Summary

* example of dispersion relation: plasma in ionosphere

① use Newton's laws (Maxwell's eqns, etc.) to write down a wave equation
The wave eqn itself comes from mechanics or E & M, i.e. it's just an equation of motion of a system that you could have written down by yourself before even taking this course.

② find the dispersion relation
Here, you do use what you've learned in this course: you plug in a normal mode ("trial wave function")
$Ae^{i(kx-\omega t)}$

When you're done taking derivatives and cancelling terms, you're left with the dispersion relation $\omega(k)$

③ compute $v_{phase} = \frac{\omega}{k}$ and $v_{group} = \frac{d\omega}{dk}$

④ for a plasma, find that $\omega<\omega_p$ leads to imaginary k, so the wave decays into the medium, i.e. it doesn't propagate, i.e. it reflects

* example using Fourier analysis to derive $v_{group} = \frac{d\omega}{dk}$

and compute analytically how a pulse spreads as it propagates, knowing only the dispersion relation and the initial pulse shape

$$\boxed{\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{-i\omega t}\, d\omega \\ \tilde{F}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \end{aligned}}$$

← inverse Fourier transform

← Fourier transform

Next time: recap, more examples, answer questions