Physics 15c (Hoffman)
Lecture #1
Thursday, September 2, 2010

Reading: H & L 1.1–1.5
or
Georgi chapter 1
or
Morin section 1.1

Goals for this course:

① Gain a physical intuition and mathematical understanding of waves, in preparation for quantum mechanics, in which all particles are represented as waves.
→ put math you probably already know into a better physical context

* → ② Fourier transforms: develop fluency thinking in the time domain and frequency domain, and transitioning between the two.

③ Practical optics

* This will be the most important and broadly applicable concept from this course.

Waves are everywhere!
- water waves
- sound
- earthquakes
- light
- radio waves
- microwaves
- human waves

2 features which define a "wave": oscillation & propagation

Wave is an oscillation at each space point, which propagates through space: motion of one point causes the next adjacent point to move.

More mathematically (à la Georgi):
most waves we study have several features in common

① linearity (mathematically: equation of motion is linear in the coordinate x, so solutions add
physically: usually means restoring force is directly proportional to displacement)

② time translation invariance
(mathematically: if $x(t)$ is a solution, so is $x(t+a)$
physically: laws of physics and physical parameters of system don't change with time, so we can start the clock at any time)

③ local interactions (each point is influenced only by nearby points)

④ space translation invariance (analogous to time translation invariance)

2 types of waves:

① mechanical waves
- require some "medium" which is actually wiggling back & forth (oscillating) at each space point
- e.g. air, earth, water, people

② "medium-less" waves
- no medium! propagate through vacuum!
- e.g. electromagnetic waves (light)
  probability waves (quantum mechanics)

Modern (20th century) physics has 2 pillars:

1. Relativity - inspired by the absoluteness of the speed of light $c = 3\times10^8$ m/s → electromagnetic waves
2. Quantum mechanics - everything (even particles!) are described by wave functions

→ Need solid understanding of waves in order to understand QM and all of modern physics.

We'll start with mechanical waves, because they're more physically intuitive, but math is the same → good lead-in to E&M waves → in turn lead into modern waves

Overall plan of attack:

- SHO (damped, driven) → use as tool for big math review
  "simple harmonic oscillator"
- coupled oscillators (2, 3, 4, ... N large)
- take $N \to \infty$ & study waves in continuous media (e.g. strings)
- phase/group velocities, dispersion
- Fourier transforms
- wave propagation, energy transfer, reflection
- sound waves & music
- E & M waves
- optics (interference, diffraction, lenses, etc.)
- modern waves

Ready, set, go!

Goals for Today:

Understand the **relevance** and **importance** of linearity:

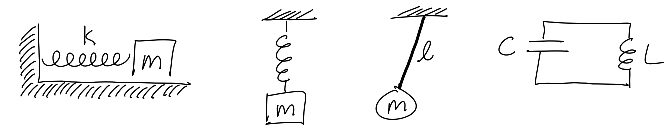
→ why almost every physical system can be described by a simple harmonic oscillator

→ how this enables us to view oscillations in the frequency domain using Fourier transforms

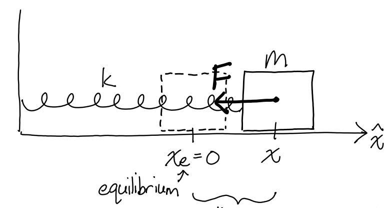
1. simple harmonic oscillators
2. linearity of the real world
3. Fourier transforms

Simple Harmonic Oscillators

Examples:



Let's look at 1st one in more detail:



displacement $\Delta x = x - x_e = x$

$$F = -kx$$

Newton's 2nd law: $\sum F = ma = m\dfrac{d^2x}{dt^2}$

$$\Rightarrow -kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \Big\}$$ equation of motion
= differential equation
→ need to solve for a function $x(t)$

What kind of function has 2nd derivative = negative const. × self?
Ans: sine or cosine!

Try $x(t) = A\cos(\omega t)$, plug in to see if it works:

$$\downarrow$$
$$\frac{dx}{dt} = -\omega A\sin(\omega t)$$
$$\downarrow$$
$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t)$$

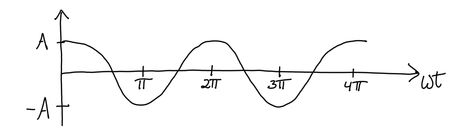
$$\Rightarrow -\omega^2 A\cos(\omega t) = -\frac{k}{m}A\cos(\omega t)$$

This eqn is true (i.e. our guessed soln works) iff $\omega^2 = \frac{k}{m}$

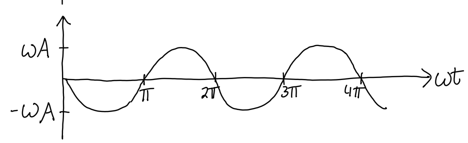
$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

So we have:

$x(t) = A\cos(\omega t)$



$v(t) = -\omega A\sin(\omega t)$



Note $x(t)$ and $v(t)$ are out-of-phase by $90° = \pi/2$.

Oscillation repeats itself at $\omega t = 2\pi$

$$\Rightarrow \text{period } T = \frac{2\pi}{\omega}$$

ω is the natural "angular frequency"
= how much the phase of the cosine advances per unit time
(unit is radian/second)

$f = \frac{1}{T} = \frac{\omega}{2\pi}$ is the natural frequency
(sometimes also called ν = Greek letter "nu")

= cycles per second
(unit is s^{-1} = Hertz)

Energy conservation:

potential energy: spring stores energy when stretched or compressed

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$$

kinetic energy: moving mass has kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t)$$

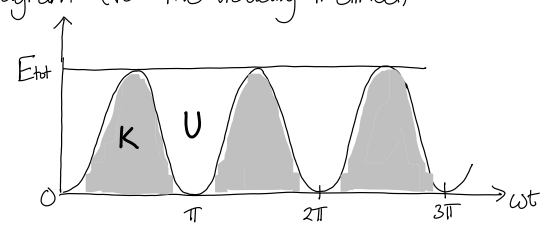
recall $\omega^2 = \frac{k}{m}$
$\Rightarrow m\omega^2 = k$

$$K = \frac{1}{2}kA^2\sin^2(\omega t)$$

Therefore, $E_{tot} = U + K$

$$E = \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}kA^2\sin^2(\omega t)$$
$$= \frac{1}{2}kA^2\left[\cos^2(\omega t) + \sin^2(\omega t)\right] = \frac{1}{2}kA^2 = \text{const}$$

Energy diagram (for the visually inclined):



Energy "sloshes" between the spring & mass, keeping total const.
Note that the energy sloshing frequency is twice the frequency of motion of the mass (because $\cos^2$ has twice the frequency of cos).

General Solution

Our solution $x(t) = A\cos(\omega t)$ cannot be the most general solution. For example, what if we started the mass with some velocity at time $t=0$? Then $v(t) = -\omega A\sin(\omega t)$ would not be correct. So we need a more general soln to meet general initial conditions. We know that both sine & cosine satisfy the differential eqn (2^{nd} deriv = $-$const $\times$ self)

$\Rightarrow$ try $x(t) = A\cos(\omega t) + B\sin(\omega t)$

Note that:

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$ is a linear, 2^{nd} order differential equation

linear: means no terms like x^2 or $x\frac{dx}{dt}$

2^{nd} order: means highest order term has exactly 2 derivatives

Linearity: If $x(t)$ is a solution, then $Ax(t)$ will be too. If $x(t)$ and $y(t)$ are solutions, then $x(t)+y(t)$ will be too.

$\Rightarrow x(t) = A\cos(\omega t) + B\sin(\omega t)$ is definitely a solution!

But how do we know that's all? How do we know there's no mystery function $\xi(t)$ such that the most general solution would be

$x(t) = A\cos(\omega t) + B\sin(\omega t) + C\xi(t)$??

Another (more mathematical) way to ask the same question is: how do we know that $\cos(\omega t)$ and $\sin(\omega t)$ form a "complete" set of solutions that "span" the solution space?

A vector analogy: $\hat{x}$ and $\hat{y}$ together form a complete basis for $\mathbb{R}^2$. So do $\hat{x}$ and $\hat{x}+\hat{y}$, but the latter basis is not orthogonal. However, neither $\{\hat{x}, \hat{y}\}$ nor $\{\hat{x}, \hat{x}+\hat{y}\}$ form a complete basis for $\mathbb{R}^3$.

2^{nd} order: An n^{th} order linear differential eqn has exactly n "linearly independent" solutions. *

* For now, just accept this as a mathematical fact. We will come back to it.

"Linearly independent" means we can't make a linear combo of them sum to zero unless all coeffs = zero, i.e. $A\cos(\omega t) + B\sin(\omega t) \neq 0$ unless $A = B = 0$.

So, mathematically, we know we have only 2 linearly independent solns to our eqn of motion, so $x(t) = A\cos(\omega t) + B\sin(\omega t)$ is the most general soln!

Physical argument for 2 solutions:

Physical system is completely specified by only 2 initial conditions x_0 and v_0
(What about a_0? Initial acceleration is determined by the force which is determined by the initial position, via Hooke's law.)
From these 2 conditions x_0 and v_0, nature knows what to do, in other words motion is completely deterministic given x_0 and v_0.

So we can use x_0 and v_0 to solve for A and B:

1. $x(t) = A\cos(\omega t) + B\sin(\omega t)$
   $x(t=0) = A = x_0$

2. $v(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$
   $v(t=0) = \omega B = v_0$
   $\Rightarrow B = v_0/\omega$

$\Rightarrow x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$

But if we had a 3^{rd} function $C\xi(t)$ then we couldn't use these 2 initial conditions to solve for all 3 A, B, and C
→ system would not be determined
→ Contradicts the fact that nature does know what to do given only 2 initial conditions

Why do we care?

Ok, great, we solved a mass on a spring.
We don't encounter masses & springs much in daily life...
or do we?

It turns out almost any system you will ever encounter in the real world can be well approximated as a mass-on-spring.

Why? Consider a particle moving in one dimension, in an arbitrary potential $U(x)$.

Particle feels a force $F(x) = -\frac{dU}{dx}$ (conservative force)

Suppose this particle has found its equilibrium position (as most things do - try dropping your book off your desk!)

At a point of equilibrium x_e, F vanishes:

$$F = -\frac{d}{dx}U(x)\Big|_{x=x_e} = -U'(x_e) = 0$$

Note that this is an energy extremum. For stable equilibrium, we also require that this is an energy minimum, so the function $U(x)$ should be concave up at this point, so the second derivative should be positive:

$$\frac{d^2}{dx^2}U(x)\Big|_{x=x_e} > 0$$

Let's consider small displacements of the particle near x_e:

We need to do a Taylor expansion:

$$U(x) = U(x_e) + (\boldsymbol{x} - x_e)\frac{d}{dx}U(x)\Big|_{x=x_e} + \frac{1}{2}(\boldsymbol{x} - x_e)^2\frac{d^2}{dx^2}U(x)\Big|_{x=x_e} + \dots + \frac{1}{n!}(\boldsymbol{x} - x_e)^n\frac{d^n}{dx^n}U(x)\Big|_{x=x_e}$$

(where the $\boldsymbol{x}$ dependence is **bold**; all else is constants)

$$F(x) = -\frac{dU}{dx} = \underbrace{-U'(x)\Big|_{x=x_e}} - (\boldsymbol{x} - x_e)\,U''(x)\Big|_{x=x_e} - \frac{1}{2}(\boldsymbol{x} - x_e)^2\,U'''(x)\Big|_{x=x_e} - \dots - \frac{1}{(n-1)!}(\boldsymbol{x} - x_e)^n\frac{d^n}{dx^n}U(x)\Big|_{x=x_e}$$

But we already know that this term vanishes, b/c x is an extremum of $U(x)$, because particle is in equilibrium.

Now the 2^{nd} term in the force looks like Hooke's law, with $k = U''(x)\big|_{x=x_e}$

[Note again: 2^{nd} deriv of $U(x)$ must be >0 to look like Hooke's law.]

Better yet, if x is close to x_e, then $x - x_e$ is small, so higher order terms become negligible!

What's the point? We started with any arbitrary potential energy, but we derived the equation of motion for a mass-on-spring as long as the displacement from equilibrium is small enough.

⇒ Every physically stable object can be approximated as a harmonic oscillator
(except for a few pathological cases in which 2^{nd} deriv vanishes or is not defined - e.g. see Georgi section 1.8)

Consequences of linearity: **Fourier transforms**

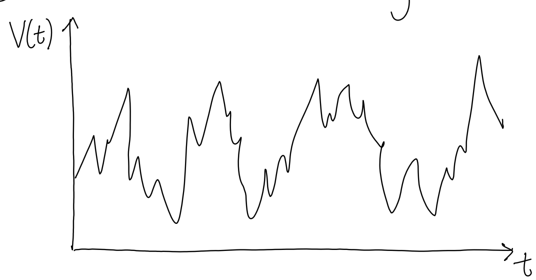
We have considered a simple system with a single mass moving in one dimension, i.e. a single "degree of freedom". This led us to a general solution which is the sum of 2 solutions of the same frequency ω.

Had we considered a more complicated system with multiple degrees of freedom (i.e. multiple masses and/or multiple directions of motion, each of which with their own, approximately linear restoring force), we would have found a more complicated general solution which would be the sum of multiple sines and cosines of different frequencies.

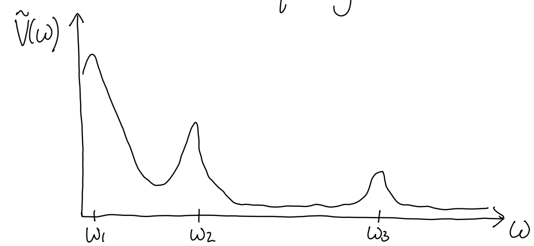
The point to emphasize here is that even our more complicated system is still <u>linear</u>; the different frequency solutions can still be <u>summed</u> to arrive at the general solution.

Now suppose we are observing some single-variable output of such a system, as a function of time.

For example, suppose that the system is the entire environment of a laboratory, and the single variable observable is the voltage noise in an experimental measurement we are trying to perform. Suppose we are watching the time evolution of this voltage on an oscilliscope and sadly scratching our heads, wondering where all this unwanted noise is coming from.



A useful way to proceed is to "Fourier transform" our noise trace. This means to find a way to represent it not as a function of time, but rather as a function of frequency - in other words, recognizing that it must be the linear combination of many oscillations at many frequencies, so representing it as a sort of "histogram" of how much oscillation we have at each frequency.



Conceptually, we will find something like

$$V(t) = \sum_{\omega} \underbrace{\tilde{V}(\omega)}_{\text{this is a measure of the amplitude of oscillation at each frequency } \omega} \cos(\omega t)$$

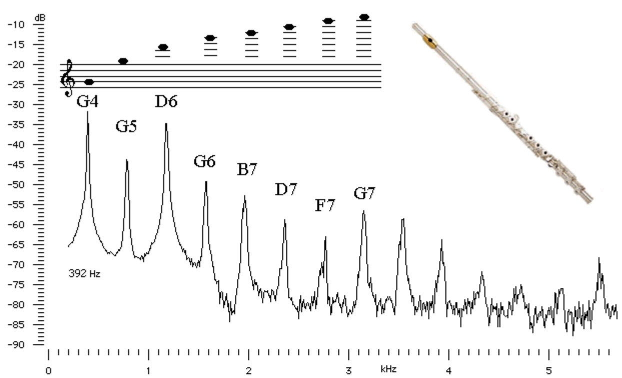
(There is a simple mathematical algorithm for getting from $V(t)$ to $\tilde{V}(\omega)$ but we'll leave those details for later and focus on the main concepts for now.)

We might then find, for example, that our noise is dominated by a few frequencies ω_1, ω_2, and ω_3, which would have been hard to pick out by eye from the raw $V(t)$. Then we can seach our lab for offending equipment that is vibrating at those 3 frequencies, and shut down that equipment.

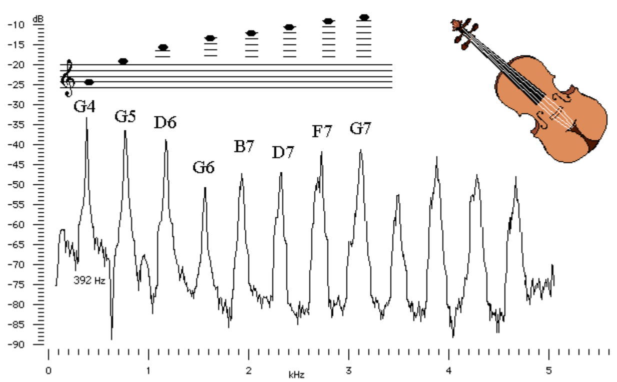
For another example, consider musical instruments. When you hear an instrument playing the note G, your ears are detecting a vibration of the air at frequency 392 Hz.

However, you know that a G played on a flute sounds very different from a G played on a violin. Why? Because what you hear is a time trace composed of the main 392 Hz oscillation, and also many other, smaller oscillations, because flutes and violins are complicated systems with many degrees of freedom.

Flute



Violin



http://www.intmath.com/Fourier-series/6_Line-spectrum.php

We can therefore easily compare a flute and violin by looking at the frequency spectra of their noise output when playing the note G. The frequency spectrum tells us how much oscillation there is at each frequency, and is the Fourier transform of the air displacement vs. time that our ears directly feel.

Fourier spectroscopy is a widely applicable, tremendously useful tool - maybe the most useful thing you will learn in this course. It is the consequence of the linearity of waves: the fact that the equations of motions are linear and therefore the various solutions may simply be summed to arrive at the most general solution.

Summary

Simple harmonic oscillator
→ eqn of motion is linear, 2^{nd} order differential equation
has 2 complete & linearly independent solutions

Almost everything in the world can be approximated by SHO
→ math review of Taylor series

Linearity → allows decomposition of time trace
via Fourier transform
into a frequency spectrum

Next time:
* damped oscillator
* review of complex numbers
* more on linear differential eqns

Reading for next time: H & L 1.6
or
Georgi 2.1
or
Morin section 1.2