Physics 15c (Hoffman)
Lecture # 18
Thurs, November 4, 2010

Midterm Review

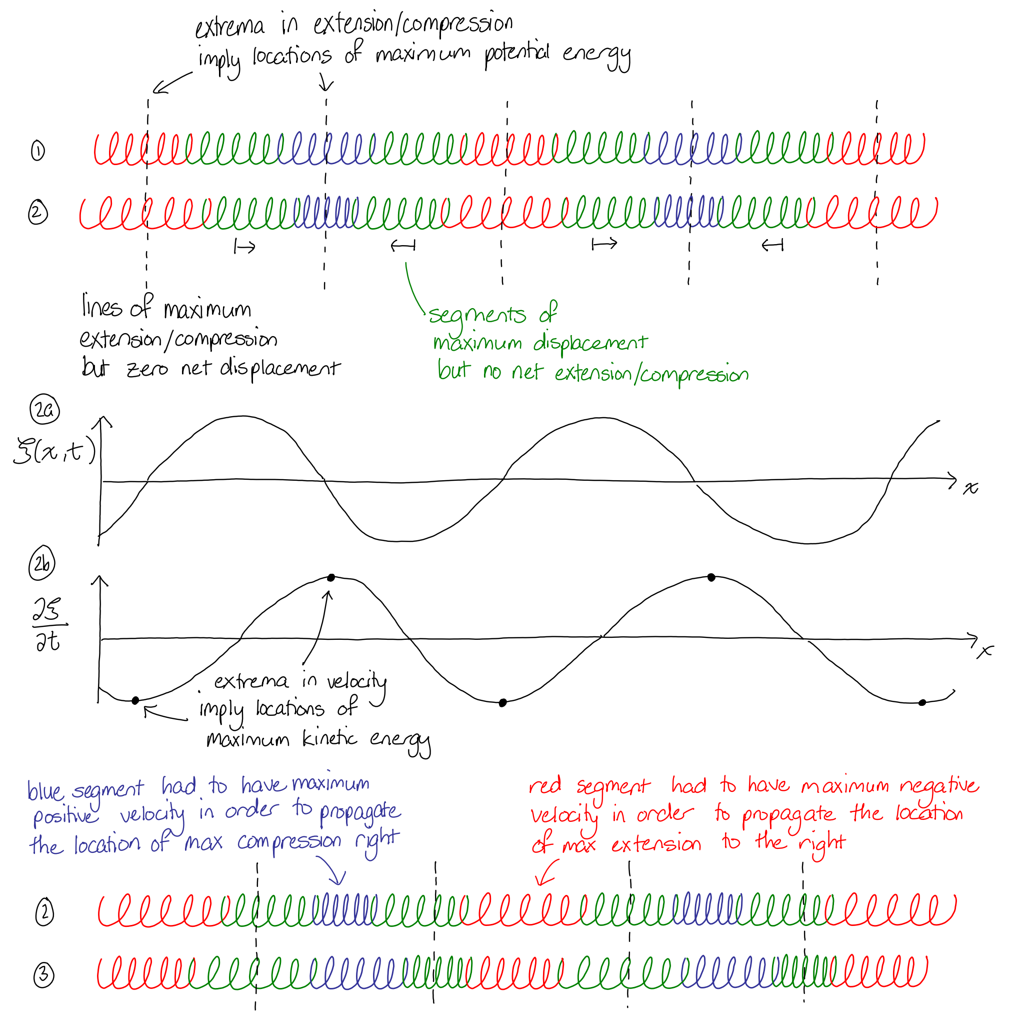
Lecture #9 : Sound

* sound = longitudinal pressure wave
* dynamic range = ratio of largest detectable input to smallest detectable input
* human ear perceives wave intensity = power / area
* decibels = $10 \log\left(\frac{I}{I_0}\right)$ where $I_0 = 10^{-12}\ \mathrm{W/m^2}$ = minimum detectable intensity
* average power carried by a wave: $\langle P \rangle = \frac{1}{2} Z \omega^2 \xi_0^2$
* impedance of a medium: $Z = \sqrt{E \rho_e}$
* equation of state: $P \propto \frac{1}{V^\gamma}$ where $\gamma = \frac{\text{specific heat at constant } P}{\text{specific heat at constant } V} = \frac{C_P}{C_V} > 1$
* bulk modulus $= B = \frac{\text{stress}}{\text{strain}} = \frac{-\Delta P}{\Delta V/V} = \gamma P$
* $v_{\text{sound}} = \sqrt{\frac{B}{\rho}}$; $Z_0 = \sqrt{B\rho}$
* intensity $I = \frac{1}{2} Z_0 \omega^2 \xi_0^2 \Rightarrow \xi_0 = \frac{1}{\omega}\sqrt{\frac{2I}{Z_0}}$

$$\Delta p = \sqrt{2 I Z_0}$$

Illustration of the counter-intuitive fact that the locations of maximum kinetic energy and maximum potential energy are at the same place, at the same time, so energy travels in packets.

① Shows the unstretched spring for reference
② Shows a snapshot of the spring at an instant in time t_0
③ Shows a snapshot of the spring at time $t_0 + T/4$, for reference



Lecture #10: Doppler shift & Shock waves

* Doppler shift (non-relativistic):

$$f' = \frac{(c_w + v_m) - v_o}{(c_w + v_m) - v_s} f_o$$

* Doppler shift (relativistic):

$$f' = \sqrt{\frac{c+v}{c-v}}\, f_o$$

* Shock waves
  * accumulation of energy in wavefront or cone when source exceeds the speed of sound
  * can get light shock waves too! = "Cerenkov radiation"
* Mach cone angle: $\theta = \sin^{-1}\left(\frac{c_w}{v_s}\right)$

Questions:

(a) Why didn't we take into account relativistic velocity addition?

Answer: There aren't 2 velocities being added.

moving source $\xrightarrow{\quad} v$ • stationary observer

Clock runs slowly here frequency $f' = \frac{1}{\Delta t'}$ in the moving source frame.

Stationary observer sees longer time intervals $\Delta t > \Delta t'$. Quantitatively, $\Delta t = \gamma \Delta t'$ where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

This is just pure time dilation so far, no velocities yet...

Now we get to the velocities part.

Let's plant ourselves firmly in the observer's stationary frame. In this frame:

- Δt is the time between the emissions of light crests
- c is the speed of light
- v is the speed of the source

- $c\Delta t$ is how far one crest moves towards us before the next one is emitted.
  (Note: there is some relativistic velocity addition hidden in this statement. If we had used Gallilean velocity addition we would say that the light crest would be moving towards us at $(v+c)$, akin to throwing a ball from a moving car – the total velocity of the ball is $\underset{\text{w.r.t. world}}{v_{ball}} \approx \underset{\text{(ball w.r.t. car)}}{v_{throw}} + \underset{\text{w.r.t. world}}{v_{car}}$ assuming non-relativistic speeds.

  But the light crest travels towards us at speed c, regardless of the speed of the object which emitted it.)
- $v\Delta t$ is how far the source moves towards us in the time between emissions of consecutive light crests
- Now, we're going to add 2 distances (not 2 velocities). $c\Delta t$ and $v\Delta t$ are each distances, well-defined in our stationary frame. We can therefore say that the distance between the first light pulse and the second one, at the instant the second one is emitted, is $c\Delta t - v\Delta t$
- The time it takes for light to traverse this distance is
  $$\Delta T = \frac{1}{c}(c\Delta t - v\Delta t) = \frac{c-v}{c}\Delta t = \frac{c-v}{c}\gamma\Delta t'$$
- The time between the arrival of subsequent crests to a fixed location is
  $$f = \frac{1}{\Delta T} = \frac{c}{c-v}\frac{1}{\gamma} f' = \frac{1}{1-\frac{v}{c}} \cdot \sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}\, f' = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\, f' = \sqrt{\frac{c+v}{c-v}}\, f'$$

(b) Why are the "regular" and "relativistic" Doppler effects multiplicative?

Answer: The time dilation is really an independent effect. The (dilated) time between crest emissions is just a constant factor which carries all the way through all of our distance additions, so that γ just ends up being a multiplicative factor in the end.

Lecture #11: Standing waves, Boundary conditions

* one-dimensional standing waves

$$\underbrace{A\cos(kx-\omega t) + A\cos(-kx-\omega t)}_{\text{2 opposite traveling waves}} = 2A\underbrace{\cos(kx)}_{\substack{\text{space part}\\\text{(with nodes)}}}\underbrace{\cos(\omega t)}_{\substack{\text{oscillating}\\\text{part}}}$$

→ subject to boundary conditions:
fixed end → node
free end → anti-node (maximum)

* pipe with open + closed ends

$$f = \frac{n+\frac{1}{2}}{2L}v_{sound} \Rightarrow f = \frac{v_{sound}}{4L}, \frac{3v_{sound}}{4L}, \frac{5v_{sound}}{4L}, \frac{7v_{sound}}{4L}, \ldots$$

odd harmonics

* pipe with 2 open ends

$$f = \frac{n}{2L}v_{sound} \Rightarrow f = \frac{v_{sound}}{2L}, \frac{v_{sound}}{L}, \frac{3v_{sound}}{2L}, \frac{2v_{sound}}{L}, \ldots$$

all harmonics

* Transverse wave on a string: $\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{\rho_e}\frac{\partial^2 \xi}{\partial x^2}$

Lecture #12: Reflection, Multi-Dim Waves & FTs

* Reflection:

qualitative: fixed vs. free ends
fixed → reflected up-side-down
free → reflected right-side-up

quantitative: R and T determined by impedance
R → reflection coefficient
T → transmission coefficient

$$\xi_R = \frac{Z_1 - Z_2}{Z_1 + Z_2}\xi_I \qquad \xi_T = \frac{2Z_1}{Z_1+Z_2}\xi_I$$

$$Z = \text{impedance} = \frac{\text{force}}{\text{velocity}}$$

* multi-dim waves: $\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

where $\vec{k}$ = wavevector = direction of propagation
dispersion relation: $\omega = c_w|\vec{k}|$
(infinite # of $\vec{k}$'s for each ω)

* multi-dim Fourier transforms:

2-dim
$$\begin{cases} f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)e^{i(k_x x + k_y y)}dk_x dk_y \\ \tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(k_x x + k_y y)}dx\,dy \end{cases}$$

general d
$$\begin{cases} f(\vec{x}) = \int \tilde{F}(\vec{k})e^{i\vec{k}\cdot\vec{x}}d^d k \\ F(\vec{k}) = \frac{1}{(2\pi)^d}\int f(x,y)e^{-i\vec{k}\cdot\vec{x}}d^d x \end{cases}$$

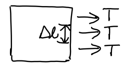
Lecture # 13: Drum Modes

* normal modes on a drum: $\omega = \pi \sqrt{\frac{T_\ell}{\rho_A}} \sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}$

(not even harmonics!)

Example: Suppose we have a square drum head, clamped on all 4 sides, with dimension L.

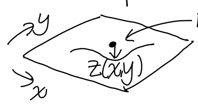
Tension is T_ℓ (in units of force/length):



$T_\ell = T/\Delta\ell$

Mass density is ρ_A (in units of mass/length^2)

At time $t=0$, we displace the center of the drum by a distance d down. The membrane will stretch in some smooth way, giving us a displacement $z(x,y)$



let's put the origin in the center: $(x,y)=(0,0)$ here

The exact functional form of $z(x,y)$ will come from solving Laplace's eqn. → may be a mess, but we do know that the center must have zero slope → $z(x,y)$ is Σ cosines and that

$$z(-\tfrac{L}{2}, y) = z(x, \tfrac{L}{2}) = z(\tfrac{L}{2}, y) = z(x, -\tfrac{L}{2}) = 0$$

⇒ cosines must have wavelengths s.t. odd # of $\frac{1}{4}\lambda$ fit between 0 and $\frac{L}{2}$

$$\frac{2n+1}{4}\lambda = \frac{L}{2} \Rightarrow k = \frac{2\pi}{\lambda} = 2\pi \cdot \frac{2n+1}{4} \cdot \frac{2}{L} = (2n+1)\frac{\pi}{L}$$

$$\Rightarrow z(x,y,t=0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m} \cos\left(\frac{(2n+1)\pi}{L}x\right)\cos\left(\frac{(2m+1)\pi}{L}y\right)$$

But how does this evolve with time?

$$\omega = \sqrt{\frac{T_\ell}{\rho_A}}\,|\vec{k}| \; ; \quad |\vec{k}| = \sqrt{k_x^2 + k_y^2} = \frac{\pi}{L}\left((2n+1)^2 + (2m+1)^2\right)^{1/2}$$

$$z(x,y,t) = \sum_{n,m} a_{n,m} \cos\left(\frac{(2n+1)\pi}{L}x\right)\cos\left(\frac{(2m+1)\pi}{L}y\right)\cos\left(\sqrt{\frac{T_\ell}{\rho_A}}\frac{\pi}{L}\sqrt{(2n+1)^2+(2m+1)^2}\,t\right)$$

And how do we get $a_{n,m}$?

$$a_{n,m} = \left(\frac{2}{L}\right)^2 \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \; z(x,y) \cos\left(\frac{(2n+1)\pi}{L}x\right)\cos\left(\frac{(2m+1)\pi}{L}y\right)$$

Lecture #14: LC transmission line

* LC transmission line wave equation:

$$\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\Delta x}{L}\frac{\Delta x}{C}\frac{\partial^2 V(x,t)}{\partial x^2} \qquad \text{(same for } q, I\text{)}$$

* impedance: $Z = \sqrt{L/C}$

* wave speed: $c_w = \sqrt{\frac{\Delta x}{L}\frac{\Delta x}{C}}$

* Ampere's Law: $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$

* Gauss' Law: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_0}$

* speed of light: $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

* energy: $\langle P \rangle = \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} V_0 I_0$

* momentum: $\langle F \rangle = \left\langle \frac{dp}{dt} \right\rangle = \frac{1}{2}\frac{V_0 I_0}{c_w}$

The intensity of sunlight hitting the Earth is about $I_{\text{sun}} = 1300\,\text{W/m}^2$.
If sunlight of this intensity strikes a perfect absorber, what pressure does it exert?
If sunlight of this intensity strikes a perfect reflector, what pressure does it exert?

Poynting vector: $\vec{S}$ = energy transfer per unit time per unit cross-sectional area = Intensity

What we want is the pressure, which is the momentum transfer per unit time per unit cross-sectional area (recall $F = \frac{dp}{dt}$).

For light (photons), $\underset{\substack{\uparrow \\ \text{energy}}}{E} = \hbar\omega$ and $\underset{\substack{\uparrow \\ \text{momentum}}}{p} = \hbar k$

So we can use the dispersion relation $\omega = ck$ to simply relate energy to momentum: $E = cp$

So the pressure is $\frac{\vec{S}}{c}$.

For a perfect absorber, we have pressure

$$P_{absorb} = \frac{1300\ \text{W/m}^2}{3\times10^8\ \text{m/s}} = 4.3\times10^{-6}\ \text{Pa} = 4.3\times10^{-11}\ \text{atm}$$

For a perfect reflector, the momentum transfer is twice as large, because the surface must re-emit the photons with equal & opposite momentum.

$$P_{reflect} = 2P_{absorb} = 8.6\times10^{-6}\ \text{Pa} = 8.6\times10^{-11}\ \text{atm}$$

Lecture #15: EM Waves

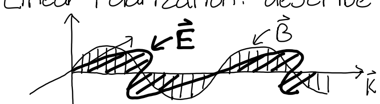
Gauss' Law — No name

$$\boxed{\begin{array}{ll} ① \ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & ③ \ \vec{\nabla}\cdot\vec{B} = 0 \\ ② \ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & ④ \ \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J} \end{array}}$$

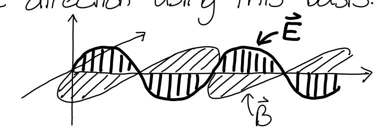
Faraday's Law — Maxwell's correction — Ampere's Law

$\oint\vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B$

* $\vec{E}$ and $\vec{B}$ are transverse waves and perpendicular to each other.
* Figure out $\vec{E}$, then use $\hat{k}\times\vec{E} = \vec{B}$ to find $\vec{B}$
* Intensity (power/area) of EM waves: $I = \frac{1}{2}c\varepsilon_0 E_0^2$
* Polarization = direction of $\vec{E}$ for EM waves
* Linear Polarization: describe any $\vec{E}$ direction using this basis:

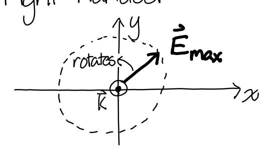


and

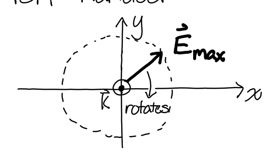


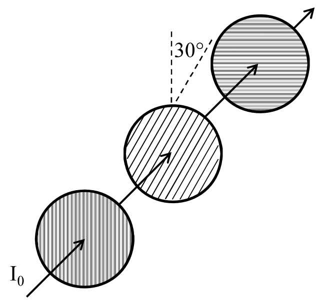
* Malus' law: $I_{out} = I_{in}\cos^2\theta$ (θ = angle between 2 polarizers)
* Circular Polarization: direction of max $\vec{E}$ rotates around $\vec{k}$-axis

right-handed:



left-handed:





(1) A plane wave of unpolarized light is normally incident on a series of linear polarizers as shown above, with the arrows indicating its propagation direction. The first polarizer has its easy axis oriented vertically (transmits vertically polarized light), the second at 30° with respect to vertical, and the third horiztonally. If the unpolarized input light intensity is I_0, what is the output intensity?

(a) $\frac{1}{64}I_0$
(b) $\frac{3}{64}I_0$
(c) $\frac{3}{32}I_0$
(d) $\frac{1}{8}I_0$
(e) $\frac{3}{16}I_0$
(f) $\frac{1}{4}I_0$
(g) $\frac{\sqrt{3}}{8}I_0$
(h) $\frac{\sqrt{3}}{4}I_0$

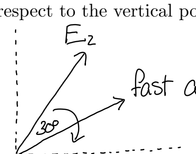
After the first polarizer, $\frac{1}{2}$ of I_0 is eliminated.

$$\frac{E_3}{E_1} = (\cos 30°)(\cos 60°) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4}$$

$$\frac{I_3}{I_0} = \frac{1}{2}\left(\frac{E_3}{E_1}\right)^2 = \frac{3}{32}$$

(2) Suppose you have a half-wave plate, which is designed to introduce a $\pi/2$ phase shift between waves polarized along its two perpendicular axes. This wave plate is placed between the second and third polarizers in the above picture. If the faster axis of the wave plate (i.e., the axis with smaller n) is aligned at 60° (clockwise) with respect to the vertical polarizer, what is the new output intensity of the whole setup?

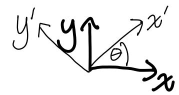
(a) 0 (no light at output)
(b) $\frac{3}{32}I_0$
(c) $\frac{1}{8}I_0$
(d) $\frac{3}{16}I_0$
(e) $\frac{27}{128}I_0$
(f) $\frac{3}{8}I_0$
(g) $\frac{1}{2}I_0$
(h) $\frac{\sqrt{3}}{2}I_0$

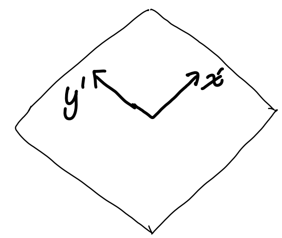


Action of $\frac{\lambda}{2}$ plate is to reflect the component of $\vec{E}_2$ which was $\perp$ to its axis, across its axis. So now it's perfectly aligned with the final horizontal polarizer.

$$\frac{I_3}{I_0} = \frac{1}{2}(\cos 30°)^2 = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{8}$$

Polarization caveat:





Linear polarizer with "easy axis" along x'

(i.e. should pass light along x' axis)

Suppose we have incoming light polarized along y' axis.

Temptation:

Incoming light: $E_0\hat{y}'e^{i(kz-\omega t)} = E_0(-\sin\theta\hat{x} + \cos\theta\hat{y})e^{i(kz-\omega t)}$

Looks like polarizer should pass the fraction of each of these that's aligned with $\hat{x}'$
$\Rightarrow$ light is transmitted!

What's wrong?

Incoming light is $\perp$ to $\hat{x}'$ $\Rightarrow$ none should be transmitted

Actually, polarizer acts as:

$$E_0(-\sin\theta\cos\theta\hat{x}' + \sin\theta\cos\theta\hat{x}')e^{i(kz-\omega t)} = 0$$

Lecture #16: EM waves in matter, reflection, refraction

* Conductors: skin depth, $1/d = \mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right)^{1/2}$

Question: where did this come from?

$$\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \quad \Rightarrow k^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$$

$\Rightarrow$ k is imaginary!

Solve for k and $\mathcal{K}$: $(k+i\mathcal{K})^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$

$k^2 - \mathcal{K}^2 = \varepsilon\mu\omega^2$; $2k\mathcal{K} = \mu\sigma\omega$

$k^2 - \left(\frac{\mu\sigma\omega}{2k}\right)^2 = \varepsilon\mu\omega^2$ $\qquad$ $\left(\frac{\mu\sigma\omega}{2\mathcal{K}}\right)^2 - \mathcal{K}^2 = \varepsilon\mu\omega^2$

$k^4 - \varepsilon\mu\omega^2 k^2 - \frac{1}{4}\mu^2\sigma^2\omega^2 = 0$ $\qquad$ $\mathcal{K}^4 + \varepsilon\mu\omega^2\mathcal{K}^2 - \frac{1}{4}\mu^2\sigma^2\omega^2 = 0$

$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right)^{1/2}$ $\qquad$ $\mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right)^{1/2}$

When $\omega \to$ small, then $\frac{\varepsilon\omega}{\sigma} \ll 1$

$$\mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{\sigma}{\varepsilon\omega}\left(\sqrt{1+\left(\frac{\varepsilon\omega}{\sigma}\right)^2} - \frac{\varepsilon\omega}{\sigma}\right)\right]^{1/2}$$

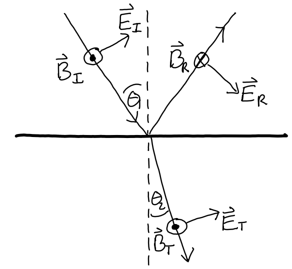
$$= \sqrt{\frac{\mu\sigma\omega}{2}}\Big[1 + \underbrace{\tfrac{1}{2}\left(\tfrac{\varepsilon\omega}{\sigma}\right)^2 - \tfrac{\varepsilon\omega}{\sigma}}_{\text{small}}\Big]^{1/2} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$$

→ $\mathcal{K}$ grows with frequency
→ at low frequency, $\mathcal{K} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$
→ skin depth gets shorter as $\omega\uparrow$
→ high f is stopped by thin metal

* Insulators: replace $\varepsilon_0 \to \varepsilon$, $\mu_0 \to \mu$ $\Rightarrow c_w = \frac{c}{n}$ where $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$
* Huygen's principle: draw successive wavefronts using point sources on previous wavefront
* Snell's law of refraction: $n_1\sin\theta_1 = n_2\sin\theta_2$
* Total internal reflection: $\theta_c = \sin^{-1}(n_1/n_2)$ [where $n_2 > n_1$]

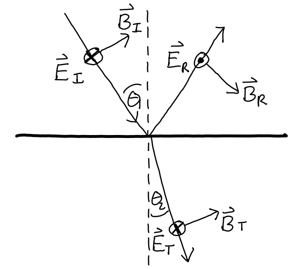
* Reflection & refraction @ non-normal incidence → Fresnel coeffs

Vertical Polarization



$$E_R = \frac{\alpha-\beta}{\alpha+\beta}E_I; \quad E_T = \frac{2}{\alpha+\beta}E_I$$

Horizontal Polarization



$$E_R = \frac{\alpha\beta-1}{\alpha\beta+1}E_I; \quad E_T = \frac{2}{\alpha\beta+1}E_I$$

[where $\alpha = \cos\theta_2/\cos\theta_1$ and $\beta = Z_1/Z_2$]

* Brewster's angle: angle at which vertically polarized light is not reflected: $\tan\theta_B = \frac{n_2}{n_1}$