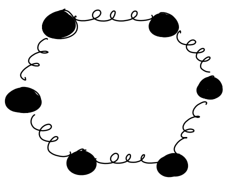
Question: how do we get from the mathematical imaginary solutions to the real physical solutions? and why do we need imaginary solutions at all - what do they mean, anyhow?

Answer: Imaginary solutions are a useful computational tool, which can dramatically reduce the number of pages of algebra to solve a problem. But they're only an intermediate step between the real statement of the physical problem and the real solution to the physical problem. So they're only effective if we know how to get from the intermediate imaginary step back to the real solutions at the end.

Note: I can't really give you any physical intuition for the imaginary solutions, because they are just that, imaginary. They're just a computational tool. All I can say is that an imaginary solution $e^{i\omega t}$ does correspond to a real solution oscillating with angular frequency ω. But we do need to understand the details of what I mean by "corresponds".

Example: Let's look back at your homework problem:



N=6 beads on springs

You could have written down a 6×6 K matrix, and diagonalized it to find the motion. This is a horrible algebraic mess. So instead you wrote down the symmetry matrix S, and without too much work, found the N=6 eigenvalues:

$$\beta_n = e^{i\frac{2\pi}{N}\cdot n} \qquad \text{for } n = 1 \ldots 6$$

and the corresponding N=6 eigenvectors:

$$b^{(n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot n} \\ e^{i\frac{2\pi}{N}\cdot 2n} \\ e^{i\frac{2\pi}{N}\cdot 3n} \\ \vdots \\ e^{i\frac{2\pi}{N}\cdot Nn} \end{pmatrix} \qquad \text{for } n = 1 \ldots 6$$

We know (because K and S commute, i.e. KS = SK) that these N=6 $b^{(n)}$'s must also be eigenvectors of our K matrix.

But the problem asked for the real eigenvectors of the K matrix. And these 6 $b^{(n)}$'s are patently imaginary.

So what do we do?

Practical answer: Take the real and imaginary parts of our $b^{(n)}$'s.

This is all you really need to remember.

But to alleviate some more confusion, lets look in more detail to see why this practical strategy works

Fact: $b^{(n)}$'s are eigenvectors of real K matrix with real eigenvalues $-\omega_n^2$

Let's prove 2 simple and relevant theorems:

Thm: If v is an eigenvector of a real matrix M with a real eigenvalue c, then v^* (complex conjugate of v) is also an eigenvector of M with the same eigenvalue c.

Proof: $Mv^* = (Mv)^* = (cv)^* = cv^*$

can pull M inside the complex conjugate operator because M is real, so $M^* = M$

can pull c outside the complex conjugate operator because c is real, so $c^* = c$

$$Mv^* = cv^* \Rightarrow v^* \text{ is an eigenvector of } M \text{ with eigenvalue } c$$

<u>Consequences</u> of this theorem to our mass-spring problem:

If $b^{(n)}$ is an eigenvector of K, then $b^{(n)*}$ must also be an eigenvector of K.

But we already know all $N=6$ eigenvectors of K.

So for each $b^{(n)}$, $b^{(n)*}$ must also be an eigenvector. Some of the $b^{(n)}$'s must be complex conjugates of each other!

Note that for $n = N/2 = 3$ and $n = N = 6$, the $b^{(n)}$'s are already real, so they are trivially their own complex conjugates.

$$b^{(3)} = b^{(3)*} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad b^{(6)} = b^{(6)*} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For other values of n, we have

$$b^{(n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot n} \\ e^{i\frac{2\pi}{N}\cdot 2n} \\ \vdots \\ e^{i\frac{2\pi}{N}\cdot Nn} \end{pmatrix}$$

$$b^{(N-n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot(N-n)} \\ e^{i\frac{2\pi}{N}(2N-2n)} \\ \vdots \\ e^{i\frac{2\pi}{N}(N^2-Nn)} \end{pmatrix} = \begin{pmatrix} e^{i\frac{2\pi}{N}N} e^{i\frac{2\pi}{N}(-n)} \\ e^{i\frac{2\pi}{N}2N} e^{i\frac{2\pi}{N}(-2n)} \\ \vdots \\ e^{i\frac{2\pi}{N}N^2} e^{i\frac{2\pi}{N}(-Nn)} \end{pmatrix} = \begin{pmatrix} e^{2\pi i} e^{-i\frac{2\pi}{N}n} \\ e^{4\pi i} e^{-i\frac{2\pi}{N}\cdot 2n} \\ \vdots \\ e^{N\pi i} e^{-i\frac{2\pi}{N}\cdot Nn} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{2\pi}{N}n} \\ e^{-i\frac{2\pi}{N}2n} \\ \vdots \\ e^{-i\frac{2\pi}{N}Nn} \end{pmatrix} = b^{(n)*}$$

So in general, $b^{(n)} = b^{(N-n)*}$.

The $b^{(n)}$'s have complex eigenvalues of matrix S, so the above theorem does not apply. But the $b^{(n)}$'s have real eigenvalues of matrix K (because these are the real frequencies of the system), so the above theorem does apply.

So $b^{(n)} = b^{(N-n)*}$ and these 2 eigenvectors share the same eigenvalue of K, with value $-\omega_n^2$.

<u>Theorem</u> If 2 eigenvectors v_1 and v_2 of matrix M share an eigenvalue c, then any linear combination of v_1 and v_2 is also an eigenvector of M with the same eigenvalue c. (Note: this second theorem does not rely on M and c being real.)

<u>Proof</u>: $M(av_1 + bv_2) = aMv_1 + bMv_2 = acv_1 + bcv_2 = c(av_1 + bv_2)$
$\Rightarrow av_1 + bv_2$ is an eigenvector of M with eigenvalue c.

(Note is this true for any general eigenvectors v_1 and v_2 of M? No, v_1 and v_2 must share the eigenvalue c.)

<u>Consequences</u> of this theorem to our mass-spring problem:

We already know from the 1st theorem that $b^{(n)}$ and $b^{(n)*}$ are both eigenvectors of the real K, sharing the same real eigenvalue $-\omega_n^2$.

$\Rightarrow$ any linear combo of $b^{(n)}$ and $b^{(n)*}$ is also an eigenvector of K

$\Rightarrow$ in particular, $\dfrac{b^{(n)} + b^{(n)*}}{2}$ and $\dfrac{b^{(n)} - b^{(n)*}}{2i}$ are eigenvectors of K

$\Rightarrow$ We get back to our original practical answer: take the Re and Im parts of each $b^{(n)}$.

Mathematically, what facts did we have to rely on to get to this point? Only: K is real and $-\omega_n^2$ is real

$\Rightarrow$ Quite generally, for any real world problem in which you come up with some set of imaginary eigenvectors, you can just take the Re and Im parts.