Physics 15c (Hoffman)
Lecture #5
Thurs, Sept 16, 2010

Reading: H&L 2.1–2.5, 4.1–4.3, 13.1–13.4
or Georgi 5.1 & 5.3
or Morin 2.4–3.3

Math: Fourier Analysis

Physics: Continuous longitudinal waves

Last time:

Coupled oscillators → coupled linear differential equations

Normal mode = solution in which all masses oscillate at the same frequency ω, and with a specific amplitude ratio

$$x(t) = \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\substack{\text{amplitude ratio} \\ =\text{spatial dependence}}} \underbrace{e^{i\omega t}}_{\substack{\text{oscillating part} \\ =\text{time dependence}}}$$

For N degrees of freedom, we have N normal modes

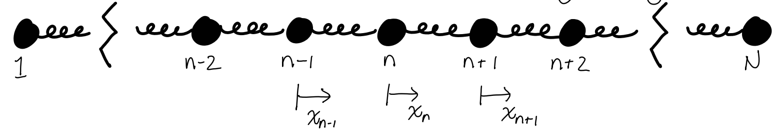
Most general solution = sum of normal modes

$$x_{tot}(t) = \sum_{i=1}^{N} x^{(i)}(t) = \sum_{i=1}^{N} \left\{ A_i \begin{pmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix} e^{+i\omega_i t} + B_i \begin{pmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix} e^{-i\omega_i t} \right\}$$

Can find normal modes by:
* brute force
* physical intuition
* linear algebra
* orthogonality
* commuting symmetry matrix

Goals for today:
* Take $N \to \infty$ ⇒ continuous wave equation
* Fourier analysis: any periodic function can be written as a sum of sines and cosines

Suppose that we have N masses, connected by springs:



Equation of motion for the n^{th} mass:

$$m\frac{d^2}{dt^2} x_n = -k(x_n - x_{n-1}) - k(x_n - x_{n+1})$$

$$\frac{d^2}{dt^2}\begin{pmatrix} \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \end{pmatrix} = \frac{1}{m}\begin{pmatrix} \ddots & & & & & & \\ \cdots k & -2k & k & & & 0 & \\ & k & -2k & k & & & \\ & & k & -2k & k & & \\ & & & k & -2k & k & \\ & 0 & & & k & -2k & k \cdots \\ & & & & & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \end{pmatrix}$$

For any given N, we could solve this (by symmetry, or by taking the determinant of an N×N matrix)

But instead, let's just take the continuum limit $N \to \infty$

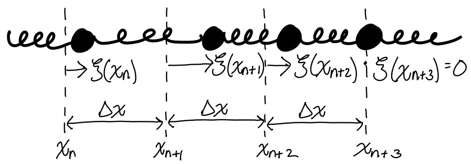
First we need to do a little bit of convenient relabeling.

Let's call the displacements of the masses ξ_n instead of x_n … this frees up the notation x_n to refer instead to the equilibrium position of the n^{th} mass.
Let the masses start out Δx apart, so $x_n - x_{n-1} = \Delta x$.

Now we might as well write ξ as a function of x, so
ξ_n = position of n^{th} mass = $\xi(x_n)$ ← equilibrium position of n^{th} mass

Now $\xi(x)$ looks like a continuous function, but it's defined only at the discrete points of interval Δx.

$\xi(x)$ will become a continuous function if $\Delta x \to 0$.



Note: $\xi_n \to \xi(x)$
$\xi_{n\pm 1} \to \xi(x \pm \Delta x)$

Rewrite the equation of motion using these new coordinates:

$$m\frac{d^2}{dt^2}\xi(x) = -k\left[\xi(x) - \xi(x-\Delta x)\right] - k\left[\xi(x) - \xi(x+\Delta x)\right]$$

Let's do some judicious multiplying and dividing by Δx:

$$\frac{m}{\Delta x}\frac{d^2}{dt^2}\xi(x) = k\,\Delta x\,\frac{\frac{\xi(x+\Delta x)-\xi(x)}{\Delta x} - \frac{\xi(x)-\xi(x-\Delta x)}{\Delta x}}{\Delta x}$$

Now make Δx smaller:

$$\lim_{\Delta x\to 0}\frac{\frac{\xi(x+\Delta x)-\xi(x)}{\Delta x} - \frac{\xi(x)-\xi(x-\Delta x)}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\xi'(x)-\xi'(x-\Delta x)}{\Delta x} = \xi''(x)$$

$$\Rightarrow \boxed{\frac{m}{\Delta x}\frac{d^2}{dt^2}\xi(x) = k\,\Delta x\,\frac{d^2\xi}{dx^2}}$$

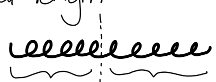
The mass term has become $\frac{m}{\Delta x}$ = linear mass density

$$\rho_\ell = \frac{m}{\Delta x} = \frac{\text{mass in kg}}{\text{length of spring in m}}$$

The spring constant has become $k\Delta x$ = "elastic modulus" $\equiv E$
(similar to Young's modulus, without the factor of cross-sectional area)

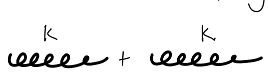
Remember that the spring constant k of a spring is inversely proportional to its natural length

e.g. if you cut a spring in half:



each half has spring constant $2k$

or double a spring:



→ will expand twice as much as each original
→ spring constant is half as big

So it makes sense to talk about a spring constant per unit of inverse length $= k/(1/L) = kL$

Modified Hooke's law:

$$F = -k\Delta L = -E\frac{\Delta L}{L}$$

↑ elastic modulus

Putting this all together:

$$\boxed{\rho_\ell \frac{\partial^2}{\partial t^2}\xi(x,t) = E\frac{\partial^2}{\partial x^2}\xi(x,t)}$$

CONTINUOUS WAVE EQUATION

↑ be a little more careful: use partial derivatives because we recognize that ξ is a function of both x and t

How do we solve this?
→ remember that we started from N coupled oscillators
→ there must have been N normal modes
(general solution is a linear combination of them)
→ now there are an infinite number of normal modes!

In the discrete case, we would expect

$$\vec{\xi} = \begin{pmatrix}\vdots\\ \xi_{n-1}\\ \xi_n\\ \xi_{n+1}\\ \vdots\end{pmatrix} = \underbrace{\begin{pmatrix}\vdots\\ a_{n-1}\\ a_n\\ a_{n+1}\\ \vdots\end{pmatrix}}_{\text{these are constants (time-independent)}} e^{i\omega t}$$

← all the time dependence is here

By extension, the normal modes for the continuous wave eqn should look like: $\xi(x,t) = a(x)e^{i\omega t}$

In other words, we should be able to factor the solution $\xi(x,t)$ into a function of x multiplied by the usual oscillator $e^{i\omega t}$.

This ability to factor the normal modes is the most important thing we've learned so far!

Now we can take our factored trial solution $\xi(x,t) = a(x)e^{i\omega t}$ and plug it in:

$$\rho_e \frac{\partial^2}{\partial t^2}\left(a(x)e^{i\omega t}\right) = E\frac{\partial^2}{\partial x^2}\left(a(x)e^{i\omega t}\right)$$

$$\rho_e\, a(x)\frac{d^2}{dt^2}e^{i\omega t} = Ee^{i\omega t}\frac{d^2}{dx^2}a(x)$$

$$-\omega^2\rho_e \cancel{e^{i\omega t}}\,a(x) = E\cancel{e^{i\omega t}}\frac{d^2}{dx^2}a(x)$$

$$\frac{d^2}{dx^2}a(x) = -\omega^2\frac{\rho_e}{E}a(x)$$

} this looks similar to a harmonic oscillator, but the derivative is in position not time

We know the solution!

$a(x) = Ae^{\pm i(\text{const})x}$ where const $= \omega\sqrt{\frac{\rho_e}{E}}$

Unfortunately this constant is called k = "wave number"
It has nothing to do with spring constant k.

wave number $= k = \frac{1}{\text{length}}$ spring constant $= k = \frac{\text{force}}{\text{length}}$

Put this together: $\xi(x,t) = Ae^{i(\omega t \pm kx)}$

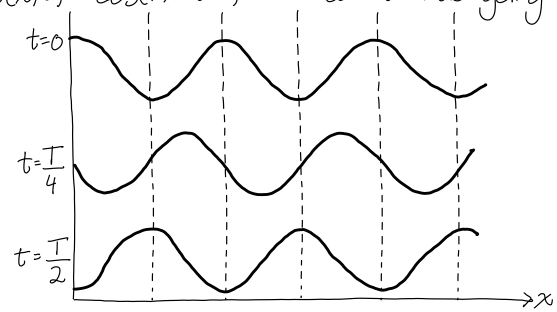
$$\boxed{\xi(x,t) = Ae^{i(kx\pm\omega t)}}$$

(A is in general complex; flip sign in exponent to match standard convention)

The equation has 2 solutions (+ and −) for any arbitrary value of ω.
→ Let's look at them separately

Recall: we are trying to describe a <u>real</u> mass-spring system
→ we can always change our solution basis to sines & cosines
→ let's consider just the cos part for simplicity
(adding in the sine part won't change the solution qualitatively but will just give us an extra degree of freedom which allows us to change the phase to match initial conditions & spatial boundary conditions)

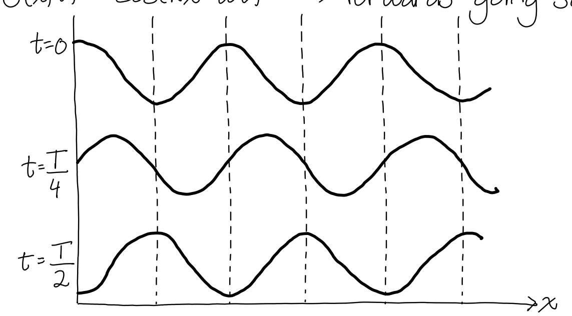
① $\xi(x,t) = \cos(kx+\omega t)$ → backwards-going solution



wave travels left!
wavelength is given by $\underbrace{k\lambda = 2\pi}_{\text{condition for periodicity of cosine}}$

$\Rightarrow k = \frac{2\pi}{\lambda}$ = "wave number"

② $\xi(x,t) = \cos(kx-\omega t)$ → forwards-going solution



wave travels right!

Wave velocity is given by the condition: $kx \pm \omega t = \text{const}$

$\Rightarrow x = \frac{\text{const} \mp \omega t}{k} \Rightarrow \frac{dx}{dt} = \mp\frac{\omega}{k}$

Another way to think of this: wave travels λ in T

$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$

In our wave equation:

$$k = \omega\sqrt{\frac{\rho_\ell}{E}} \quad \Rightarrow \frac{dx}{dt} = \mp v_{wave} = \mp\sqrt{\frac{E}{\rho_\ell}} \begin{array}{l} \leftarrow \text{elastic modulus} \\ \leftarrow \text{linear mass density} \end{array}$$

k ← wave #, ω ← frequency

Velocity v_{wave} is determined by the physical properties (E and ρ_ℓ) of the transmission line

→ doesn't depend on ω (same for all normal modes)

→ such a medium (mass + spring) is called "non-dispersive"
(later, we will find more complicated media that are dispersive, i.e. the velocity will depend on ω)

→ also, velocity doesn't depend on initial conditions

Exercise.

(a) Write down the expression for the velocity of the mass at nominal position x and time t for a forward-going normal mode of amplitude A.

(b) How does the velocity from (a) compare to the wave velocity?

(c) What is the condition for $v_{mass} < v_{wave}$?
(What happens if the condition is violated?)

Fourier Analysis

How do we create waves?
We excite one end of the transmission line and the disturbance propagates.

Let's consider only right-traveling solutions for the moment:

$$\xi(x,t) = Ae^{i(kx-\omega t)}$$

Actual solution must be real → change basis

$$\rightarrow \xi(x,t) = a\cos(kx-\omega t) + b\sin(kx-\omega t)$$

Notice: ω can be any real number
(unlike in N masses + springs where we have only N frequencies)

ω = any real # → there are an infinite number of normal modes, as we would expect from our continuum limit

But there is one constraint: for a given ω, k is fixed

$$k = \omega\sqrt{\frac{\rho_\ell}{E}} \Big\} \text{ this is called the "dispersion relation"}$$

For this simple case of masses & springs, dispersion is linear

→ v_{wave} is constant for all modes

→ jargon: the system is "dispersion-less"

(More complicated systems → non-linear dispersion relation → "dispersive")

OK, so the most general solution must be a linear combination of all (infinitely many!) normal modes

$$\xi(x,t) = \sum_\omega \left[a_\omega\cos(kx-\omega t) + b_\omega\sin(kx-\omega t)\right]$$

This is called a **Fourier series**

Any periodic function $f(t)$ with period T can be expressed by a Fourier series:

$$f(t) = a_0 + a_1 \cos\left(\frac{2\pi t}{T}\right) + a_2 \cos\left(\frac{4\pi t}{T}\right) + \ldots$$
$$+ b_1 \sin\left(\frac{2\pi t}{T}\right) + b_2 \sin\left(\frac{4\pi t}{T}\right) + \ldots$$
$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constants found by

$$\left.\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\,dt \\ a_n &= \frac{1}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right)dt \\ b_n &= \frac{1}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)dt \end{aligned}\right\} \text{checking these will be HW}$$

So we could drive our transmission line with a repetitive but non-sinusoidal pattern:

Fourier expansion gives us a way to describe the motion in term of sines and cosines (which are mathematically easy to work with)

$$f(t, x=0) = \cancel{a_0} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$

ignore a_0 because it doesn't make waves

write $\omega_n = \frac{2\pi n}{T}$

then $f(t, x=0) = \sum_{n=1}^{\infty}\left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right]$

Now each sinusoidal drive term results in a propagating wave:

$$\cos(\omega_n t) \rightarrow \cos\left[\omega_n\left(t - \frac{x}{v_{wave}}\right)\right]$$
$$\sin(\omega_n t) \rightarrow \sin\left[\omega_n\left(t - \frac{x}{v_{wave}}\right)\right]$$

In other words, the repetitive pattern of movement $f(t)$ generates a wave: $\xi(x,t) = f\left(t - \frac{x}{v_{wave}}\right)$

which propagates at velocity v_{wave}

We can choose the period of T freely - even infinite!

↳ although this makes the calculation of a_n, b_n impractical

What if $f(t)$ is not repetitive?

→ you can drive the transmission line with arbitrary pattern $f(t)$ and generate a wave $\xi(x,t) = f\left(t - \frac{x}{v_{wave}}\right)$

In fact, the wave eqn: $\rho_e \frac{\partial^2}{\partial t^2}\xi(x,t) = E\frac{\partial^2}{\partial x^2}\xi(x,t)$

is satisfied by any function of the form $\xi(x,t) = f(x \pm v_{wave} t)$

$$\begin{aligned} \text{LHS} &= \rho_e \frac{\partial^2}{\partial t^2} f(x \pm v_{wave} t) \\ &= \rho_e v_{wave}^2 f''(x \pm v_{wave} t) \\ &= E f''(x \pm v_{wave} t) \end{aligned} \qquad \begin{aligned} \text{RHS} &= E\frac{\partial^2}{\partial x^2} f(x \pm v_{wave} t) \\ &= E f''(x \pm v_{wave} t) \end{aligned}$$

Seems too simple after all this work!

How did we get here?

① N masses + springs turn into continuous transmission line as $N \to \infty$ (So N eqns of motion → single continuous wave eqn)

② Linear algebra tells us that there are N normal modes that can be factored: $x(t) = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} e^{i\omega t}$

(So we factor our continuous solution: $\xi(x,t) = a(x)e^{i\omega t}$)

③ Using this factoring tip, we can solve the wave eqn to get $\xi(x,t) = Ae^{i(kx \pm \omega t)}$

(So we find sine waves which travel @ constant velocity)

Linearity of wave equation

↳ ④ Fourier series allow us to break arbitrary functions into sines & cosines, each of which we already know is a solution.

So, since we know all solutions travel at the same velocity, we can run this argument backwards and say that any sum of sines and cosines with the correct velocity is a solution! **Any** waveform with correct v_{wave} is a solution!

(Then we confirmed this by plugging in $\xi(x,t) = f(x \pm v_{wave} t)$.)

Subtleties...

These conclusions appear stupidly simple:
We can generate any wave and it will travel.

But there are non-trivial **assumptions**
① linearity
② constant velocity for all normal modes
↓
not always true!
e.g. water waves
light passing through glass } these are "dispersive" media
↓
we will come back to this

Summary:

* studied (finally!) continuous waves
* started from mass-spring transmission line
* found wave eqn & its normal modes:

$$\rho_L \frac{\partial^2}{\partial t^2} \xi(x,t) = E \frac{\partial^2}{\partial x^2} \xi(x,t) \longrightarrow \xi(x,t) = Ae^{i(kx \pm \omega t)}$$

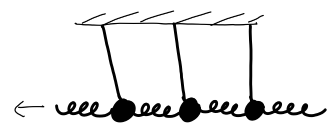
* solutions represent waves traveling @ constant $v_{wave} = \sqrt{E/\rho_L}$
* use Fourier series to show that waveform can be arbitrary

Next time:

* beats
* phase velocity vs. group velocity
* information carried by waves

Reading: H & L 2.6-2.7
Georgi 5.2
Morin 2.1.4, 6.3
} a little tough b/c we're going in a slightly different order than the textbooks

Sneak Preview Exercise:



Take the continuum limit to find the eqn of motion for this system.