Physics 15c (Hoffman)
Lecture #12
Thurs, Oct 14, 2010

Reading for today: H&L ch6
or Georgi 9.1 & 11.1
or Morin ch 4

Waves on strings & reflections

Last time

* Shock waves: accumulation of energy in wavefront or cone when source exceeds the speed of sound

* one-dimensional standing waves

$$A\cos(kx-\omega t) + A\cos(-kx-\omega t) = 2A\cos(kx)\cos(\omega t)$$

2 opposite traveling waves; space part (with nodes); oscillating part

→ subject to boundary conditions:
fixed end → node
free end → anti-node (maximum)

* pipe with open + closed ends

$$f = \frac{n+\frac{1}{2}}{2L} v_{sound} \Rightarrow f = \frac{v_{sound}}{4L}, \frac{3v_{sound}}{4L}, \frac{5v_{sound}}{4L}, \frac{7v_{sound}}{4L}, \ldots$$

odd harmonics

* pipe with 2 open ends

$$f = \frac{n}{2L} v_{sound} \Rightarrow f = \frac{v_{sound}}{2L}, \frac{v_{sound}}{L}, \frac{3v_{sound}}{2L}, \frac{2v_{sound}}{L}, \ldots$$

all harmonics

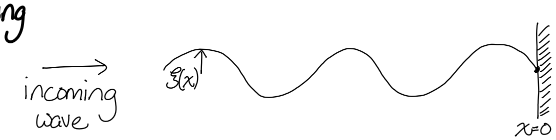
* Transverse wave on a string: $\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{\rho_L}\frac{\partial^2 \xi}{\partial x^2}$

Goals For Today

* reflection
  → qualitative: fixed vs. free ends
  → quantitative: need impedance
* 2-dimensional waves: $\vec{k}$ is wavevector (upgraded from wavenumber)
* derivation of wave eqn on a drum head
* boundary conditions in multi-dimensions

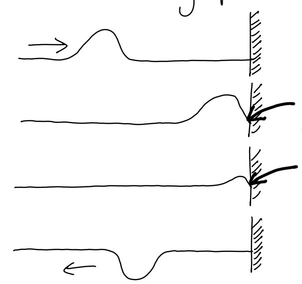
Reflection on a string

HARD wall



one end of string is tied to fixed point, end cannot move
$\Rightarrow \xi(x=0) = 0$

Consider an incoming pulse:



pulse travels toward HARD wall

pulse on string pulls UP on wall

Newton's 3rd law: wall pulls DOWN on string

pulse is reflected will OPPOSITE polarity

Quantitatively: consider a single frequency component

total wave = incoming + outgoing

$$\xi_{tot}(x,t) = \xi_0 \sin(kx-\omega t) + A\sin(-kx-\omega t)$$

reflection gives - sign to x term
amplitude yet to be determined

$$\xi(x=0,t) = \xi_0 \sin(-\omega t) + A\sin(-\omega t) = 0$$

$$\Rightarrow A = -\xi_0$$

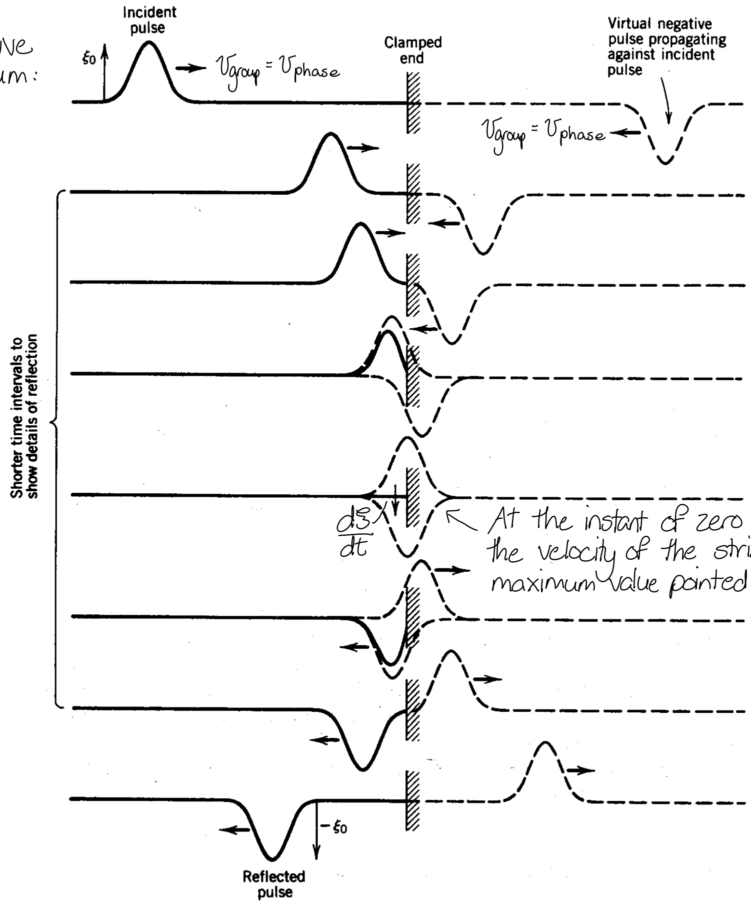
Any pulse is a superposition of single-frequency components so any pulse reflects with equal but opposite amplitude.

Reflection in time & space:

For a perfectly hard wall, no energy is lost at reflection
⇒ pulse should travel backwards the same as forwards.

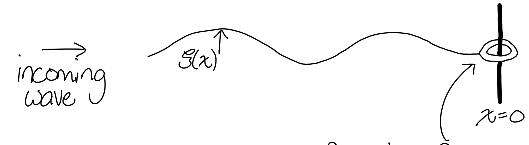
Can think of this as an imaginary pulse of opposite amplitude traveling towards the wall from the right; the actual pulse on the string will be the sum of the unreflected real + imaginary pulses.

Non-dispersive medium:



At the instant of zero amplitude, the velocity of the string has a maximum value pointed down.

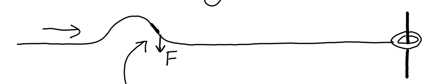
SOFT wall



end of string free to move up & down, no transverse force on end

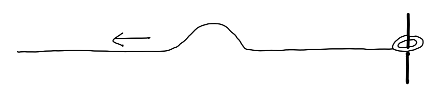
$$\Rightarrow \left.\frac{d\xi}{dx}\right|_{x=0} = 0$$

Consider an incoming pulse:



an increment on the leading edge feels a force pulling down from the lower string ahead of it

when the leading edge reaches the soft wall, there is no longer any restoring force from the right, so the string continues to rise until it reaches double its amplitude, and the restoring force from the left pulls it down again



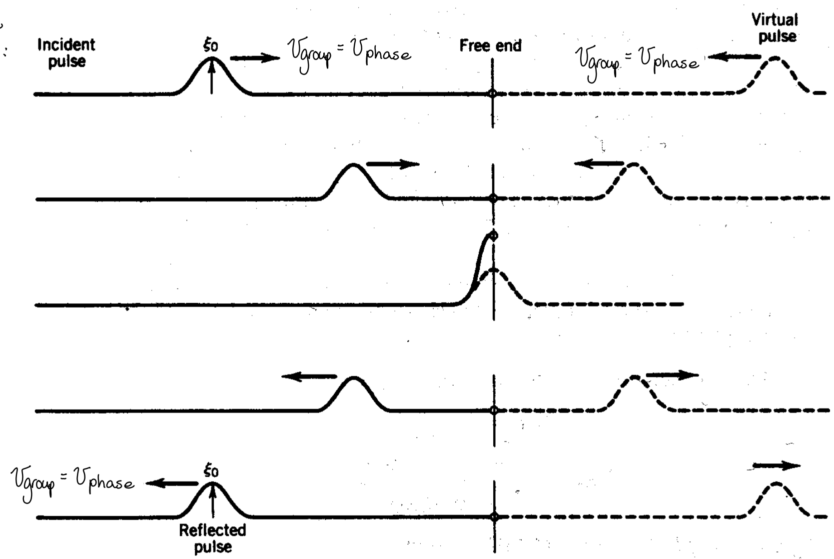
Quiz: Derive the amplitude of the reflected pulse quantitatively.

Reflection in time & space:

For a perfectly soft wall, no energy is lost at reflection
$\Rightarrow$ pulse should travel backwards the same as forwards.

Can think of this as an imaginary pulse of same amplitude traveling towards the wall from the right; the actual pulse on the string will be the sum of the unreflected real + imaginary pulses.

Non-dispersive medium:



(This is like the method of image charges you used in physics 15b.)

Reflection summary so far:

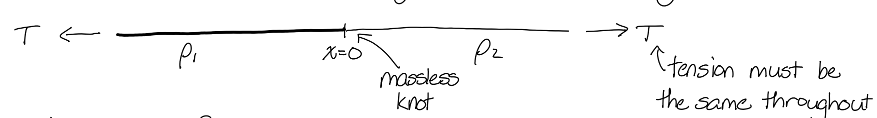
Hard boundary, fixed end $\rightarrow$ total reflection
inverted polarity

Soft boundary, free end $\rightarrow$ total reflection
same polarity

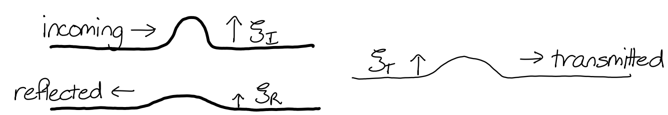
Intermediate cases???

In practice, complete reflection rarely happens.
Usually, some of the wave's energy is absorbed or transmitted by the boundary, and only part of it is reflected.

Consider reflection between a lighter & heavier string:



Send in a pulse from the left:



How to find ξ_R and ξ_T in terms of ξ_I and properties of 2 strings?

① Conservation of energy: $E_I = E_R + E_T$

$$\frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_I^{\,2} = \frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_R^{\,2} + \frac{1}{2}\sqrt{\rho_2 T}\,\omega^2 \xi_T^{\,2}$$

② Boundary condition: $\xi_I + \xi_R = \xi_T$
(must have single-valued, well-defined displacement at the point of reflection!)

Combine ① and ②:

$$\sqrt{\rho_1 T}\,\xi_I^2 = \sqrt{\rho_1 T}\,\xi_R^2 + \sqrt{\rho_2 T}\,(\xi_I + \xi_R)^2$$

$$\Big\downarrow \text{ algebra}$$

$$\boxed{\xi_R = \frac{\sqrt{\rho_1 T} - \sqrt{\rho_2 T}}{\sqrt{\rho_1 T} + \sqrt{\rho_2 T}}\,\xi_I}$$ **reflected amplitude**

$$\sqrt{\rho_1 T}\,\xi_I^2 = \sqrt{\rho_1 T}\,(\xi_T - \xi_I)^2 + \sqrt{\rho_2 T}\,\xi_T^2$$

$$\Big\downarrow \text{ algebra}$$

$$\boxed{\xi_T = \frac{2\sqrt{\rho_1 T}}{\sqrt{\rho_1 T} + \sqrt{\rho_2 T}}\,\xi_I}$$ **transmitted amplitude**

$\sqrt{\rho T}$ is the impedance of the string $\equiv Z$

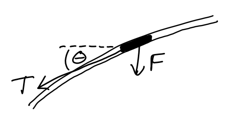
⇒ if the two strings are "impedance-matched", then there will be no reflection (complete transmission)

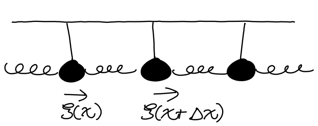
What is impedance anyhow?

$$Z = \frac{\left(\text{Force exerted by each increment on the next as the wave is propagated}\right)}{(\text{Velocity of the increment})} = \frac{F}{v}$$

Consider a traveling mechanical wave: $\xi(x,t) = \xi_0 \sin(kx - \omega t)$

→ Force is given by the space derivative





$$F = -T\sin\theta \approx -T\tan\theta \approx -T\frac{\partial \xi}{\partial x}$$

$$F = -k_s(\xi(x+\Delta x) - \xi(x)) \approx -k_s\Delta x \frac{\partial \xi}{\partial x} = -E\frac{\partial \xi}{\partial x}$$

→ Velocity is given by the time derivative: $v = \frac{\partial \xi}{\partial t}$

→ Impedance

elastic constant: T (string) or E (springs)

$$Z = \frac{F}{v} = \frac{-(\text{const})\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial t}} = \frac{-(\text{const})k\xi_0\cos(kx-\omega t)}{-\omega\xi_0\cos(kx-\omega t)}$$

$$= (\text{const})\frac{k}{\omega} = \frac{(\text{const})}{v_{phase}}$$

strings: $Z = \frac{T}{v_{phase}} = \frac{T}{\sqrt{T/\rho_\ell}} = \sqrt{T\rho_\ell}$

springs: $Z = \frac{E}{v_{phase}} =$
→ $\frac{E}{\sqrt{E/\rho_\ell}} = \sqrt{E\rho_\ell}$ (non-dispersive, no pendulums)
→ $\frac{E}{\omega c_w/\sqrt{\omega^2-\omega_0^2}} = \sqrt{E\rho_\ell}\cdot\frac{\sqrt{\omega^2-\omega_0^2}}{\omega}$ (dispersive)

In a mechanical wave of fixed frequency, the force and velocity are always proportional to each other (because time and space derivatives of sine are proportional to each other).

Non-dispersive media: Z is constant for all frequencies
→ wave pulse reflects & transmits with same shape

Dispersive media: Z depends on frequency
→ different frequency components of a pulse will reflect and transmit in different ratios

Quiz: Suppose a wave travels along a string with tension 10 N. A massless knot connects a string with density 9 g/m to a string with density 4 g/m. What fraction of the wave's energy is transmitted from the heavier string to the lighter one?

How can you maximize the energy transmission?

Alternative Derivation

Another way to think of the reflection-transmission boundary-matching conditions (which makes it more apparent why impedance is the relevant quantity):

position of increment is single-valued @ boundary $\Longleftrightarrow$ velocity of increment is single-valued @ boundary

$$\xi_I + \xi_R = \xi_T \qquad\qquad \frac{\partial \xi_I}{\partial t} + \frac{\partial \xi_R}{\partial t} = \frac{\partial \xi_T}{\partial t}$$

energy is not created or destroyed @ boundary $\Longleftrightarrow$ there can't be an infinite force on any increment @ boundary

i.e. power in = power out

$$\frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_I^2 = \frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_R^2 + \frac{1}{2}\sqrt{\rho_2 T}\,\omega^2 \xi_T^2$$

i.e. as we take the boundary increment size $\to 0$, the force on that increment must also $\to 0$, in order to keep the acceleration finite

forces on the boundary

$$F_I - F_R - F_T = 0$$

force exerted by incoming wave from the left

$+F_R$ is the force that the interface must exert to get the reflected wave to move back to the left

$+F_T$ is the force that the interface must exert to get the transmitted wave to move to the right

Newton's 3rd law says that $-F_R$ and $-F_T$ are the forces exerted by each of these waves on the interface.

OK, so the point is, we can write the reflection-transmission boundary conditions in terms of energy and amplitude...

... or we can write them in terms of force and velocity, in which case it's natural that the impedance $Z = F/v$ is the relevant quantity.

Multi-dimensional waves

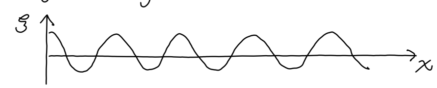
One-dim: we know how to write a wave $\xi(x,t)$:

$$\xi_0 e^{i(kx \pm \omega t)} \quad \text{or} \quad \xi_0 \sin(kx \pm \omega t)$$

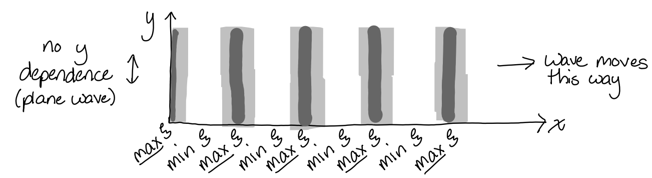
Two-dim: how to extend naturally to $\xi(x,y,t)$?

Naively, just ignore $y \to$ declare that $\xi(x,y,t)$ is constant in y

e.g. $\xi(x,y,t) = \xi e^{i(kx-\omega t)}$



look down from above:



But this can't be the whole story
→ we have to be able to describe waves that propagate in any arbitray (x,y) direction

As long as the medium is isotropic (= "the same in all directions") then the physics should not depend on direction.

Extend the 1-dim wave eqn to make it isotropic:

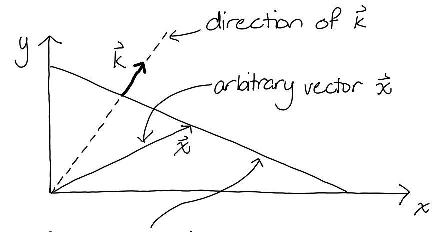
$$\frac{\partial^2 \xi(x,y,t)}{\partial t^2} = c_w^2 \left(\frac{\partial^2 \xi(x,y,t)}{\partial x^2} + \frac{\partial^2 \xi(x,y,t)}{\partial y^2} \right)$$

$$\Rightarrow \xi(x,y,t) = \xi_0 e^{i(k_x x + k_y y - \omega t)} = \xi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\Rightarrow \omega^2 = c_w^2 (k_x^2 + k_y^2) = c_w^2 |\vec{k}|^2$$

$\vec{k}$ = "wavevector" = (k_x, k_y)

$\xi(\vec{x},t) = \xi_0 e^{i(\underbrace{\vec{k}\cdot\vec{x}} - \omega t)}$ ← draw this dot product:



Note that $\vec{k}\cdot\vec{x}$ is the same for any $\vec{x}$ on this line.
Therefore, $\xi(\vec{x},t)$ is the same for all points on the line!
→ the line is a "wavefront";
→ it propagates with velocity given by $\vec{k}\cdot\vec{x} - \omega t = \text{const.}$
(The location of the wavefront at $\vec{x}$ is defined by the place where $\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ takes a particular value. So in order to find the location $\vec{x}$ where $\xi(\vec{x},t)$ continues to take that particular value as t evolves, we need the argument $\vec{k}\cdot\vec{x} - \omega t$ to stay constant. The requirement that $\vec{k}\cdot\vec{x} - \omega t$ stays constant gives an equation relating $\vec{x}$ to t which defines the velocity.)

Phase velocity:

During a time interval Δt, the wavefront will move $(\Delta x, \Delta y)$ given by:

$\vec{k}\cdot\vec{x} - \omega t = \text{const.}$

change = $k_x\Delta x + k_y \Delta y$ ↗ ↖ change = $\omega \Delta t$

these 2 changes must be equal

$$\frac{k_x}{|\vec{k}|}\frac{\Delta x}{\Delta t} + \frac{k_y}{|\vec{k}|}\frac{\Delta y}{\Delta t} = \frac{\omega}{|\vec{k}|} \equiv v_{phase}$$

$$\hat{k}\cdot\frac{d\vec{x}}{dt} = v_{phase}$$

↖ so we see that the velocity, projected onto the direction of $\vec{k}$ (i.e. $\hat{k}$) is equal to its own magnitude, v_{phase}
⇒ $\vec{v}$ of the wave is along the direction of $\vec{k}$

Quiz: plane wave

(a) Write down $\xi(\vec{x},t)$ for a plane wave of amplitude 5 cm, wavelength 20 cm traveling in the (3,4) direction with speed 10 cm/s.
(b) What is the frequency of this wave?

Fourier Transforms in $d \geq 2$

We have plane wave solutions for travel in arbitrary direction:

$$\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad \text{where} \quad \omega = c_w|\vec{k}|$$

Note: there are now **infinitely** many allowed $\vec{k}$ values for every ω. The constraint imposed by the dispersion relation $\omega = c_w|\vec{k}|$ is on the magnitude of $\vec{k}$, but any direction is allowed.

Each of these $\xi(\vec{x},t) = \xi_0{}^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with appropriate ω and $\vec{k}$ is a normal mode solution to the continuous wave equation:

$$\frac{\partial^2 \xi(\vec{x},t)}{\partial t^2} = c_w^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\xi(\vec{x},t) = c_w^2 \nabla^2 \xi(\vec{x},t)$$

As with all normal modes we have looked at, any arbitrary wave of arbitrary shape can be expressed by a linear combination of these normal modes
→ they are **complete**
→ they **span** the vector space of all possible solutions to the wave equation

Extension of Fourier transform to multi-dim:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)e^{i(k_x x + k_y y)}\,dk_x dk_y$$

$$= \int \tilde{F}(\vec{k})e^{i\vec{k}\cdot\vec{x}}\,d^2k \qquad \text{(same formula as above, just shorthand)}$$

$$\tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-i(k_x x + k_y y)}\,dx\,dy$$

$$= \frac{1}{(2\pi)^2}\int f(x,y)e^{-i\vec{k}\cdot\vec{x}}\,d^2x$$

Wave propagation from initial conditions:

If a wave has form $f(x,y)$ at $t=0$

→ then the Fourier integral can break it into components of the form $e^{i\vec{k}\cdot\vec{x}}$

→ and each component will travel as: $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ (where ω depends on $\vec{k}$ via dispersion relation)

→ so if we know the initial shape of a wave pulse $f(x,y)$ then we know how it travels for all time

Quiz: Suppose $f(x,y) = e^{-(x^2+y^2)/2}$ at $t=0$

on a medium that satisfies $\frac{\partial^2 \xi}{\partial t^2} = c_w^2 \nabla^2 \xi$

Write an expression for the full $\xi(x,y,t)$ for all $t>0$ (don't bother to explicitly compute any integrals).

Summary

* Reflection:

qualitative: fixed vs. free ends

fixed → reflected up-side-down; free → reflected right-side-up

quantitative: R and T determined by impedance

R → reflection coefficient; T → transmission coefficient

$\xi_R = \frac{Z_1 - Z_2}{Z_1 + Z_2}\xi_I$ $\quad$ $\xi_T = \frac{2Z_1}{Z_1+Z_2}\xi_I$

Z = impedance = $\frac{\text{force}}{\text{velocity}}$

* multi-dim waves: $\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

where $\vec{k}$ = wavevector = direction of propagation
dispersion relation: $\omega = c_w|\vec{k}|$
(infinite # of $\vec{k}$'s for each ω)

* multi-dim Fourier transforms:

2-dim
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$
$$\tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

general d
$$f(\vec{x}) = \int \tilde{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d^d k$$
$$F(\vec{k}) = \frac{1}{(2\pi)^d}\int f(x,y)\, e^{-i\vec{k}\cdot\vec{x}}\, d^d x$$

Next time: Drum; LC transmission line

Reading for next time: H&L 7.5, 9.1–9.4
or Georgi 11.2–11.3; 8.4
or Morin 7.1, 8.1

QUIZ Answers

Reflection at a free end:

$$\xi_{tot}(x,t) = \xi_0 \sin(kx-\omega t) + A\sin(-kx-\omega t)$$

reflection gives − sign to x term

amplitude yet to be determined

Boundary condition: $\left.\frac{\partial \xi_{tot}}{\partial x}\right|_{x=0} = 0$ (vertical force vanishes)

$$\frac{\partial \xi_{tot}(x,t)}{\partial x} = k\xi_0 \cos(kx-\omega t) - kA\cos(-kx-\omega t)$$

$$\left.\frac{\partial \xi_{tot}}{\partial x}\right|_{x=0} = k\xi_0\cos(\omega t) - kA\cos(\omega t) = 0$$

$$\Rightarrow \xi_0 = A$$

Maximize energy transmission:

$\rho_1 = 9$ g/m ; $\rho_2 = 4$ g/m ; T is same on both strings → irrelevant

$$\xi_T = \frac{2\sqrt{\rho_1 T}}{\sqrt{\rho_1 T} + \sqrt{\rho_2 T}}\xi_I = \frac{2\sqrt{9}}{\sqrt{9}+\sqrt{4}}\xi_I = \frac{6}{5}\xi_I$$

$$\frac{(\text{Energy})_T}{(\text{Energy})_I} = \frac{\frac{1}{2}\sqrt{\rho_2 T}\,\omega^2 \xi_T^2}{\frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_I^2} = \frac{2\left(\frac{6}{5}\xi_I\right)^2}{3(\xi_I)^2} = \frac{24}{25}$$

For a fixed ρ_1, we can write the ratio and maximize:

$$\frac{(\text{Energy})_T}{(\text{Energy})_I} = \frac{\frac{1}{2}\sqrt{\rho_2 T}\,\omega^2 \xi_T^2}{\frac{1}{2}\sqrt{\rho_1 T}\,\omega^2 \xi_I^2} = \frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}\left(\frac{2\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}}\right)^2 = \frac{4\sqrt{\rho_1\rho_2}}{(\sqrt{\rho_1}+\sqrt{\rho_2})^2}$$

$$\frac{d}{d\rho_2}\left(\frac{4\sqrt{\rho_1\rho_2}}{(\sqrt{\rho_1}+\sqrt{\rho_2})^2}\right) = \frac{4\sqrt{\rho_1}/(2\sqrt{\rho_2})}{(\sqrt{\rho_1}+\sqrt{\rho_2})^2} + \frac{4\sqrt{\rho_1\rho_2}(-2)}{(\sqrt{\rho_1}+\sqrt{\rho_2})^3}\cdot\frac{1}{2\sqrt{\rho_2}}$$

$$= \frac{2\sqrt{\rho_1/\rho_2}\,(\sqrt{\rho_1}+\sqrt{\rho_2}) - 4\sqrt{\rho_1}}{(\sqrt{\rho_1}+\sqrt{\rho_2})^3} = \frac{2\rho_1/\sqrt{\rho_2} + 2\sqrt{\rho_1} - 4\sqrt{\rho_1}}{(\sqrt{\rho_1}+\sqrt{\rho_2})^3}$$

$$= \frac{2\sqrt{\rho_1/\rho_2}}{(\sqrt{\rho_1}+\sqrt{\rho_2})^3}(\sqrt{\rho_1} - \sqrt{\rho_2}) = 0 \quad \Rightarrow \rho_2 = \rho_1$$

⇒ As we expect, the energy transmission is maximized if $\rho_2 = \rho_1$, i.e. there is no boundary at all, so nothing is reflected.

Plane Wave

(a) $\lambda = 20\text{ cm} = 0.2\text{ m}$

$$|\vec{k}| = \frac{2\pi}{0.2\text{m}} = 10\pi\ \text{m}^{-1}$$

$$k_x = \frac{3}{\sqrt{3^2+4^2}}|\vec{k}| = \frac{3}{5}10\pi\ \text{m}^{-1} = 6\pi\ \text{m}^{-1}$$

$$k_y = \frac{4}{\sqrt{3^2+4^2}}|\vec{k}| = \frac{4}{5}10\pi\ \text{m}^{-1} = 8\pi\ \text{m}^{-1}$$

$$v_{phase} = 10\text{ cm/s} = 0.1\text{ m/s} = \frac{\omega}{|\vec{k}|} \Rightarrow \omega = (0.1\text{ m/s})(10\pi\ \text{m}^{-1}) = \pi\ \text{s}^{-1}$$

$$\xi(\vec{x},t) = (5\text{ cm})\,e^{2\pi i(3x+4y-t/2)}$$

(b) $f = \omega/2\pi = (\pi\ \text{s}^{-1})/2\pi = 0.5$ Hz

Wave propagation from initial conditions

$$\tilde{F}(\vec{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\ e^{-(x^2+y^2)/2}\,e^{-i(k_x x + k_y y)}$$

(Note: you could do this integral by completing the square in x and y, shifting the origin, then proceeding in polar coords.)

$$\xi(x,y,t) = \int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\ \tilde{F}(\vec{k})\,e^{i(k_x x + k_y y - c_w|\vec{k}|t)}$$

(Note on the relative sign of k_x, k_y and ω: the sign choice for k_x, k_y is arbitrary as long as it is opposite for the forward & inverse FT's. Sign for ω is chosen so wave spreads out as t increases.)