Physics 15c (Hoffman)
Lecture #4
Tues, Sept 14, 2010

Reading: Georgi 3.1–3.3 and 4.1 or Morin 2.1–2.3

Math: Linear Algebra, Eigenvalues & Eigenvectors

Physics: Coupled Oscillators

Big picture road map:

* we just spent 3 lectures on single oscillators (simple, damped, driven)
  → learned to solve single linear differential equations
* today we talk about N coupled oscillators (start with $N=2$)
  → N simultaneous linear differential equations
  → need linear algebra
* next time we take the limit as $N \to \infty$, "continuum limit"
  → WAVES!

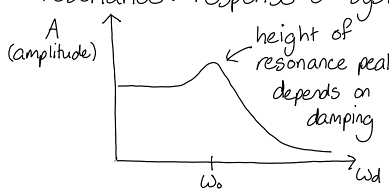
Last time (more specifically):

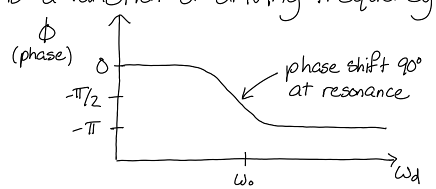
* inhomogenous linear differential equations
  → allowed us to solve for oscillator w/ driving force:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)$$

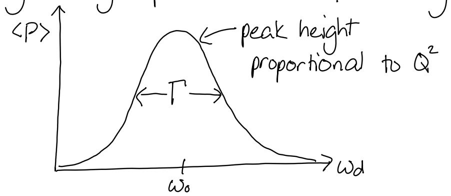
(as a specific example, we solved for an LCR circuit)

* resonance: response of system is a function of driving frequency





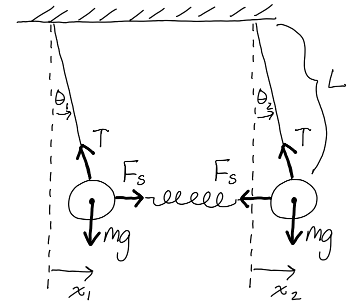
* Q = "quality factor" of oscillator $= \omega_0/\Gamma$
  = how easily it rings up at resonance, inversely proportional to damping



Goals for today:

* coupled oscillators
  → simultaneous linear differential eqns
  → linear algebra
  → symmetry
* prepare for real waves

Example: coupled pendulums



2 coupled pendulums, 3 forces on each:
$\vec{T}$ is canceled by $mg\cos\theta$
⇒ only 2 net forces on each:

* spring force F_s (along x-axis)
* net gravitational force $F_g = mg\sin\theta$ (<u>almost</u> along x-axis)

Consider only small oscillations: θ_i small $\Rightarrow x_i = L\sin\theta_i \approx L\theta_i$
⇒ consider motion only on x-axis (good approx)

① $\Sigma F = F_s - F_g = kx_2 - kx_1 - mg\sin\theta_1$
$\approx -k(x_1 - x_2) - \frac{mg}{L}x_1$

② $\Sigma F = -F_s - F_g = -kx_2 + kx_1 - mg\sin\theta_2$
$\approx -k(x_2 - x_1) - \frac{mg}{L}x_2$

Newton's 2nd law → 2 coupled linear differential eqns

$$m\frac{d^2x_1}{dt^2} = -\frac{mg}{L}x_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -\frac{mg}{L}x_2 - k(x_2 - x_1)$$

How do we solve coupled lin. diff. eqns?

* brute force
* physical intuition
* linear algebra
* orthogonality of normal modes
* commuting symmetry matrix

Brute force

① $m\frac{d^2x_1}{dt^2} = -\frac{mg}{L}x_1 - k(x_1 - x_2)$

$$\Rightarrow x_2 = \frac{1}{k}\left[m\frac{d^2x_1}{dt^2} + \left(\frac{mg}{L}+k\right)x_1\right]$$

plug in

② $m\frac{d^2x_2}{dt^2} = -\frac{mg}{L}x_2 - k(x_2 - x_1)$

$$\frac{m^2}{k}\frac{d^4x_1}{dt^2} + \frac{m}{k}\left(\frac{mg}{L}+k\right)\frac{d^2x_1}{dt^2} = -\frac{mg}{kL}\left[m\frac{d^2x_1}{dt^2} + \left(\frac{mg}{L}+k\right)x_1\right] - \left[m\frac{d^2x_1}{dt^2} + \left(\frac{mg}{L}+k\right)x_1 - kx_1\right]$$

$$\frac{m^2}{k}\frac{d^4x_1}{dt^4} + 2\left(\frac{m^2g}{kL}+m\right)\frac{d^2x_1}{dt^2} + \left(\frac{1}{k}\left(\frac{mg}{L}\right)^2 + \frac{2mg}{L}\right)x_1 = 0$$

4^{th} order homogenous linear diff eqn → plug in $x_1(t) = Ae^{\alpha t}$

$$\frac{m^2}{k}\alpha^4 Ae^{\alpha t} + 2\left(\frac{m^2g}{kL}+m\right)\alpha^2 Ae^{\alpha t} + \left(\frac{1}{k}\left(\frac{mg}{L}\right)^2 + \frac{2mg}{L}\right)Ae^{\alpha t} = 0$$

$$\alpha^4 + 2\left(\frac{g}{L}+\frac{k}{m}\right)\alpha^2 + \frac{g}{L}\left(\frac{g}{L}+2\frac{k}{m}\right) = 0$$

$$\left[\alpha^2 + \frac{g}{L}\right]\left[\alpha^2 + \frac{g}{L} + 2\frac{k}{m}\right] = 0$$

$$\Rightarrow \alpha = \pm i\sqrt{\frac{g}{L}} \quad \text{or} \quad \alpha = \pm i\sqrt{\frac{g}{L}+2\frac{k}{m}}$$

For each of these 4 solutions of α, we have a solution $x_1(t) = Ae^{\alpha t}$ which we can plug into

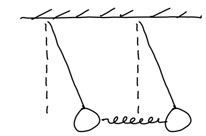
$$x_2(t) = \frac{1}{k}\left[m\frac{d^2x_1}{dt^2} + \left(\frac{mg}{L}+k\right)x_1\right]$$

But this is really a big pain!

Physical Intuition

Notice that there are 2 simple ways the pendulums can move:

①

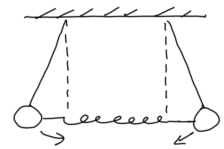


pendulums move in synch: spring is never stretched or compressed

$\Rightarrow x_1(t) = x_2(t) = A_1 e^{i\omega_1 t} + B_1 e^{-i\omega_1 t}$

where ω_1 is the same $\omega_0 = \sqrt{g/L}$ as for a single pendulum, because the spring is never stretched or compressed and therefore plays no role

②



pendulums move exactly out-of-synch: spring is stretched & compressed by double the displacement of each

$\Rightarrow x_1(t) = -x_2(t) = A_2 e^{i\omega_2 t} + B_2 e^{-i\omega_2 t}$

where $\omega_2 > \omega_1$ because the spring now adds an additional restoring force, so the system should oscillate faster

Normal Modes

These 2 different motions are called "normal modes". For a system of N coupled oscillators, it turns out there will always be N normal modes. Normal modes are a convenient way to break down complicated motion into a sum (linear combination) of a few simple motions, each of which oscillate with a single fixed frequency.

* How do we combine them?
* How do we deal with initial conditions?

The system is still linear, so the solutions just add:

$$x_1(t) = C_1\cos(\omega_1 t) + D_1\sin(\omega_1 t) + C_2\cos(\omega_2 t) + D_2\sin(\omega_2 t)$$

$$x_2(t) = C_1\cos(\omega_1 t) + D_1\sin(\omega_1 t) - C_2\cos(\omega_2 t) - D_2\sin(\omega_2 t)$$

Now we have 4 initial conditions: x_{10}, x_{20}, v_{10} and v_{20}
and we have 4 constants: C_1, D_1, C_2, and D_2
so we can just solve for C_1, D_1, C_2, D_2 in terms of $x_{10}, x_{20}, v_{10}, v_{20}$

Linear Algebra

Intuition is nice, but we need a more systematic way to solve the problem in case you don't happen to just guess the solution. We will find the normal modes using linear algebra.

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = -\dfrac{g}{L}x_1 - \dfrac{k}{m}(x_1 - x_2) \\ \dfrac{d^2x_2}{dt^2} = -\dfrac{g}{L}x_2 - \dfrac{k}{m}(x_2 - x_1) \end{cases}$$

Rewrite this system of 2 coupled eqns in matrix form:

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{g}{L} - \frac{k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad [\text{eqn } 1]$$

We are looking for a normal mode where both x_1 and x_2 oscillate with the same frequency ω:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{i\omega t} \\ a_2 e^{i\omega t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} \qquad [\text{eqn } 2]$$

Plug eqn 2 into eqn 1:

$$\underbrace{\frac{d^2}{dt^2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}}_{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}(-\omega^2) e^{i\omega t}} = \begin{pmatrix} -\frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{g}{L} - \frac{k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

Divide through by $e^{i\omega t}$ (note $|e^{i\omega t}| = 1$ always, so no fear of $\div$ by 0)

$$-\omega^2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{g}{L} - \frac{k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

call this vector $\vec{a}$

call this matrix $\overleftrightarrow{K}$

$$\Rightarrow \overleftrightarrow{K}\vec{a} = -\omega^2\vec{a}$$

Eigenvectors & Eigenvalues:

Matrix $\overleftrightarrow{K}$ times vector $\vec{a}$ = scalar constant times $\vec{a}$ itself

- → we call $\vec{a}$ an "eigenvector" of the matrix $\overleftrightarrow{K}$
- → the constant $-\omega^2$ is called the "eigenvalue"
- → in general, an $N \times N$ matrix has N eigenvectors
- → these give N normal modes

OK, but how do we solve for $\vec{a}$ and ω^2? Rewrite:

$$\overleftrightarrow{K}\vec{a} + \omega^2\vec{a} = 0$$

$$\overleftrightarrow{K}\vec{a} + \omega^2\overleftrightarrow{I}\vec{a} = 0$$

identity matrix = $\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$

$$(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I})\vec{a} = 0$$

Remember, $\vec{a}$ is the amplitude ratio for the motion of the 2 masses as they oscillate simultaneously at the single frequency ω: $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$

So we don't want the trivial $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ solution because that would just correspond to no motion.

But if $(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I})$ is an invertible matrix, then we could write:

$$(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I})^{-1}(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I})\vec{a} = 0 \Rightarrow \vec{a} = 0$$

So we need to ensure that $(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I})$ has no inverse.

→ in fact this is true if the determinant vanishes:

$$\det(\overleftrightarrow{K} + \omega^2\overleftrightarrow{I}) = 0$$

Computing the determinant:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei-hf) - b(di-gf) + c(dh-ge)$$

bigger matrices? → use Mathematica
http://downloads.fas.harvard.edu/download
Student license number: L2983-5986
Need to send email to
<mathsw@fas.harvard.edu> to get password.

Back to our example:

$$\det\begin{pmatrix} \omega^2 - \frac{g}{\ell} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \omega^2 - \frac{g}{\ell} - \frac{k}{m} \end{pmatrix} = 0$$

$$\left(\omega^2 - \frac{g}{\ell} - \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

↓ factor using difference of squares

$$\left(\omega^2 - \frac{g}{\ell}\right)\left(\omega^2 - \frac{g}{\ell} - \frac{2k}{m}\right) = 0$$

$$\Rightarrow \omega_1 = \pm\sqrt{\frac{g}{\ell}} \quad \text{and} \quad \omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

The 2 values of $-\omega^2$ are our eigenvalues of matrix $\overleftrightarrow{K}$.
→ they give us our normal mode frequencies

Now we need to find our eigenvectors
→ to tell us exactly what is moving @ these frequencies

Start back from $(\overleftrightarrow{K} + \omega^2 \overleftrightarrow{I})\vec{a} = 0$

① $\omega_1 = \sqrt{\frac{g}{\ell}}$:

$$\begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

eigenvector is only defined up to a constant
(= amplitude of that normal mode)

⇒ set $a_1 = 1$ and solve for a_2 ⇒ $a_2 = 1$

⇒ normal mode is: $x^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\pm i\omega_1 t}$

② $\omega_2 = \sqrt{\frac{g}{\ell} + \frac{2k}{m}}$:

$$\begin{pmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

⇒ set $a_1 = 1$ and solve for a_2 ⇒ $a_2 = -1$

⇒ normal mode is: $x^{(2)}(t) = \begin{pmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\pm i\omega_2 t}$

Total solution is linear combination of normal modes.

Linear algebra gives a recipe for finding normal modes
by solving an eigenvalue problem.

But what if we have 6 coupled oscillators?
→ need to take the determinant of 6×6 matrix
→ solve 6^{th} order polynomial
→ can use Mathematica
→ or can use mathematical symmetry arguments
* symmetry of $\overleftrightarrow{K}$ matrix
* commuting symmetry operator (à la Georgi, ch4)

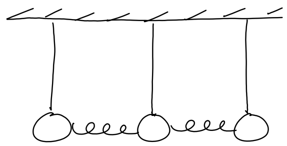
Symmetry of $\overleftrightarrow{K}$

If there is no damping, the $\overleftrightarrow{K}$ matrix is symmetric.

Physically, this follows from Newton's 3rd law: $\vec{F}_{12} = \vec{F}_{21}$
(in words: every action generates an equal & opposite reaction)
and from the fact that we are dealing with conservative potentials (no damping)
→ see Georgi p61 for more rigorous proof

Fact: the eigenvectors of a symmetric matrix are orthogonal

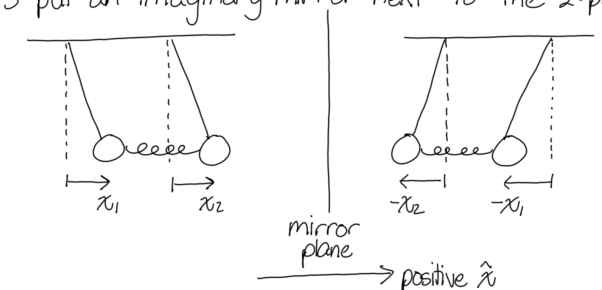
Challenge: use this fact, combined with your physical intuition, to come up with the 3 normal mode eigenvectors for:



system of
3 coupled pendulums

Commuting Symmetry Operator

Let's put an imaginary mirror next to the 2-pendulum system:



The physics of the system should be the same even if we reflect it in a mirror, i.e. the "normal mode" should be the same: same frequency, same amplitude ratio.

So if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution, then $\tilde{x}(t) = \begin{pmatrix} -x_2(t) \\ -x_1(t) \end{pmatrix}$ is too.

How can we get from $x(t)$ to $\tilde{x}(t)$?

We write down a symmetry matrix $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Check: $\tilde{x}(t) \stackrel{?}{=} Sx(t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix} = \tilde{x}(t)$ ✓

(With a little practice, you will be able to spot the symmetry in a given system and write down the matrix S.)

Real world: $\frac{d^2}{dt^2} x(t) = K x(t)$ ①

Mirror word: should be true that $\frac{d^2}{dt^2}\tilde{x}(t) = K\tilde{x}(t)$

Multiply ① by matrix S:

$$\underbrace{S\frac{d^2}{dt^2}}_{\text{constant}} x(t) = SKx(t)$$

$$\left.\begin{aligned} &\underbrace{\frac{d^2}{dt^2}\tilde{x}(t)} = SKx(t) \\ &= K\tilde{x}(t) = KSx(t) \end{aligned}\right\} \Rightarrow SKx(t) = KSx(t)$$

For scalars (numbers) the order of multiplication doesn't matter, e.g. $2\times 3 = 3\times 2 = 6$

We say that scalars "**commute**".

This is not generally true of matrices:
in general, $AB \neq BA$, so in general matrices don't commute!

e.g. $\begin{pmatrix}1&2\\3&1\end{pmatrix}\begin{pmatrix}1&4\\3&1\end{pmatrix} = \begin{pmatrix}7&6\\6&13\end{pmatrix}$

but $\begin{pmatrix}1&4\\3&1\end{pmatrix}\begin{pmatrix}1&2\\3&1\end{pmatrix} = \begin{pmatrix}13&6\\6&7\end{pmatrix}$

not the same!

But some special combinations of matrices commute!

In this 2-pendulum problem, we just found that $SK = KS$.
$\rightarrow$ the symmetry matrix commutes with the "force matrix" (for lack of a better term...)

Suppose $x(t)$ is a normal mode.
Then $\tilde{x}(t)$ is a normal mode with the same frequency.
We already know that for this system, $\omega_1 \neq \omega_2$
so if $\tilde{x}(t)$ is a normal mode with the same frequency, then it must be the <u>same</u> normal mode!

(Note: most physical systems have all distinct eigenfrequencies)

Therefore $\tilde{x}(t)$ must be just a constant times $x(t)$

$$\tilde{x}(t) = \beta x(t)$$

β ← scalar constant

$$\Rightarrow S x(t) = \beta x(t)$$

So $x(t)$ is an eigenvector of matrix S
(in addition to being an eigenvector of K, which we already knew).

[Note: although S and K share the same set of eigenvectors, they do not necessarily have the same set of eigenvalues.
In other words, it's not generally true that $\beta = -\omega^2$.]

So in order to find the normal modes, we just need to find the eigenvectors of S.

Hold on! Why did we bother? Didn't we just substitute one eigenvector problem for another? How does this help?

The reason this helps is that S is generally a much simpler matrix than K.

The symmetry matrix S is typically "sparse" (= mostly zero's) with a few 1's and -1's. So it's much simpler to take a determinant.

Furthermore, if we apply the symmetry operator two times (reflect across a mirror plane twice) then we get back to where we started.

$$S x(t) = \tilde{x}(t) = \beta x(t)$$
$$S^2 x(t) = S\tilde{x}(t) = \underbrace{x(t) = \beta^2 x(t)}_{\text{therefore, } \beta^2 = 1}$$
$$\Rightarrow \beta = \pm 1$$

We didn't even have to compute a determinant to find the eigenvalues of S!

Use $\beta_1 = -1$, $\beta_2 = +1$ to find the eigenvectors of S:

$$\beta_1 = -1 \Rightarrow \begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}-a_1\\-a_2\end{pmatrix}$$

let $a_1 = 1 \Rightarrow a_2 = a_1 = 1$

$$\beta_2 = 1 \Rightarrow \begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$$

let $a_1 = 1 \Rightarrow a_2 = -a_1 = -1$

So the eigenvectors of S are $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow$ these are the amplitude ratios for our 2 normal modes

Now we plug these back into K to find the frequencies:

$$\begin{pmatrix}-\frac{g}{l}-\frac{k}{m} & \frac{k}{m}\\ \frac{k}{m} & -\frac{g}{l}-\frac{k}{m}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-\frac{g}{l}\\-\frac{g}{l}\end{pmatrix} = \underbrace{-\frac{g}{l}}_{\text{this is the eigenvalue of } K:\ -\omega^2}\begin{pmatrix}1\\1\end{pmatrix}$$

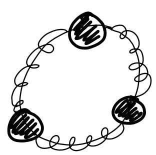
$$\Rightarrow \omega_1 = \sqrt{\frac{g}{l}} \Rightarrow x^{(1)}(t) = A_1\begin{pmatrix}1\\1\end{pmatrix}e^{i\sqrt{\frac{g}{l}}t} + B_1\begin{pmatrix}1\\1\end{pmatrix}e^{-i\sqrt{\frac{g}{l}}t}$$

$$\begin{pmatrix} -\frac{g}{l}-\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{g}{l}-\frac{k}{m} \end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix} -\frac{g}{l}-\frac{2k}{m} \\ \frac{g}{l}+\frac{2k}{m} \end{pmatrix} = -\left(\frac{g}{l}+\frac{2k}{m}\right)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow \omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}} \Rightarrow x^{(2)}(t) = A_2\begin{pmatrix}1\\-1\end{pmatrix}e^{i\sqrt{\frac{g}{l}+\frac{2k}{m}}\,t} + B_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-i\sqrt{\frac{g}{l}+\frac{2k}{m}}\,t}$$

Great! We got the solutions from symmetry!
For $N=2$, it doesn't look that much simpler.

Exercise: consider $N=3$:



3 beads of mass m on a wire, equally spaced, connected by 3 identical springs of spring constant k

(a) try to guess the normal modes from physical intuition

(b) write down the symmetry matrix S

(c) find the normal modes from S

(d) this is more time consuming, so it will appear on HW and you're not expected to do it now: find the frequencies of the normal modes by writing down the K-matrix and plugging in

Note: a, b, and c can be done without ever drawing a force diagram or writing down Newton's laws!

Summary

Coupled oscillators → coupled linear differential equations

Normal mode = solution in which all masses oscillate at the same frequency ω, and with a specific amplitude ratio

$$x(t) = \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\substack{\text{amplitude ratio} \\ =\text{spatial dependence}}} \underbrace{e^{i\omega t}}_{\substack{\text{oscillating part} \\ =\text{time dependence}}}$$

For N degrees of freedom, we have N normal modes

Most general solution = sum of normal modes

$$x_{tot}(t) = \sum_{i=1}^{N} x^{(i)}(t) = \sum_{i=1}^{N} \left\{ A_i \begin{pmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix} e^{+i\omega_i t} + B_i \begin{pmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix} e^{-i\omega_i t} \right\}$$

Can find normal modes by:

* brute force
* physical intuition
* linear algebra
* orthogonality
* commuting symmetry matrix

Next time:

Take $N \to \infty$ ⇒ continuous wave equation

Reading: H & L 2.1–2.5
or Georgi 5.1 & 5.3
or Morin 2.4