Physics 15c (Hoffman)
Lecture # 14
Thurs, Oct 21, 2010

LC Transmission Line

Last time:

* multi-dim Fourier transforms:

2-dim
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$
$$\tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

general d
$$f(\vec{x}) = \int \tilde{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d^d k$$
$$F(\vec{k}) = \frac{1}{(2\pi)^d}\int f(x,y)\, e^{-i\vec{k}\cdot\vec{x}}\, d^d x$$

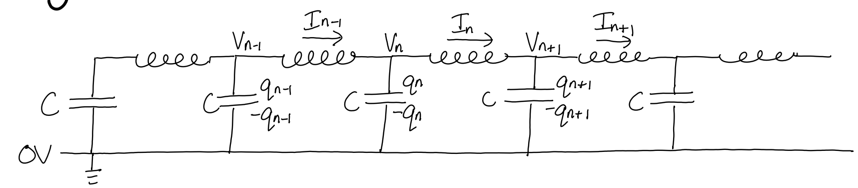
* normal modes on a drum: $\omega = \pi\sqrt{\frac{T_e}{\rho_A}}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}$

(not even harmonics!)

Goals for today:

* start from coupled LC oscillators
* find and solve the wave equation
* calculate the wave velocity
* impedance of the LC transmission line
* real example: parallel wires (review Gauss' Law, Ampere's Law)
* compute energy & momentum transmission of wave

Array of Inductors & Capacitors



[analog of masses & springs

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① voltage across C: $V_n = \frac{q_n}{C}$

② voltage across L: $V_n - V_{n+1} = L\frac{dI_n}{dt}$

[If I_n is increasing ($dI_n/dt > 0$) then Lens' law says that there is an induced emf trying to oppose the increase in I_n, so $V_n - V_{n+1}$ must be positive to overcome this emf, or else I_n would not be increasing in the first place.]

③ Charge conservation: $I_{n-1} = I_n + \frac{dq_n}{dt}$

2nd deriv. of ①: $C\frac{d^2V_n}{dt^2} = \frac{d^2q_n}{dt^2}$

1st deriv. of ③: $\frac{dI_{n-1}}{dt} - \frac{dI_n}{dt} = \frac{d^2q_n}{dt^2}$

combine $\Rightarrow C\frac{d^2V_n}{dt^2} = \frac{dI_{n-1}}{dt} - \frac{dI_n}{dt}$

plug in ② $\Rightarrow C\frac{d^2V_n}{dt^2} = \frac{1}{L}(V_{n-1} - V_n) - \frac{1}{L}(V_n - V_{n+1})$

$$\Rightarrow \frac{d^2V_n}{dt^2} = \frac{1}{LC}\left\{(V_{n+1} - V_n) - (V_n - V_{n-1})\right\}$$

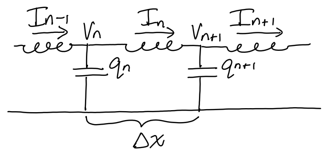
Continuum Limit

Replace indices with position:

$V_n \to V(x)$

$I_n \to I(x)$

$q_n \to q(x)$



Equation becomes:

$$\frac{\partial^2 V(x)}{\partial t^2} = \frac{1}{LC}\left\{\left[V(x+\Delta x)-V(x)\right]-\left[V(x)-V(x-\Delta x)\right]\right\}$$

$$= \frac{1}{LC}\left\{\Delta x\,\frac{\partial V(x)}{\partial x} - \Delta x\,\frac{\partial V(x-\Delta x)}{\partial x}\right\}$$

$$= \frac{(\Delta x)^2}{LC}\,\frac{\partial^2 V(x)}{\partial x^2}$$

Define $\frac{L}{\Delta x}$ = linear inductance density (Henry/m)

$\frac{C}{\Delta x}$ = linear capacitance density (Farad/m)

$$\boxed{\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\Delta x}{L}\,\frac{\Delta x}{C}\,\frac{\partial^2 V(x,t)}{\partial x^2}}$$ **Voltage wave equation**

Notice from eqn ① that $V_n \propto q_n$, so we could equally well write:

$$\boxed{\frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\Delta x}{L}\,\frac{\Delta x}{C}\,\frac{\partial^2 q(x,t)}{\partial x^2}}$$ **Charge wave equation**

(interpret in terms of charge density)

Or we could take a time derivative of both sides and write:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 q(x,t)}{\partial t^2}\right) = \frac{\Delta x}{L}\,\frac{\Delta x}{C}\,\frac{\partial}{\partial t}\left(\frac{\partial^2 q(x,t)}{\partial x^2}\right)$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial q(x,t)}{\partial t}\right) = \frac{\Delta x}{L}\,\frac{\Delta x}{C}\,\frac{\partial^2}{\partial x^2}\left(\frac{\partial q(x,t)}{\partial t}\right)$$

$$\boxed{\frac{\partial^2 I(x,t)}{\partial t^2} = \frac{\Delta x}{L}\,\frac{\Delta x}{C}\,\frac{\partial^2 I(x,t)}{\partial x^2}}$$ **Current wave equation**

Notice: Voltage and current wave eqns are the same!

Normal mode solutions

$$\begin{cases} V(x,t) = V_0 e^{i(kx \pm \omega t)} \\ I(x,t) = I_0 e^{i(kx \pm \omega t)} \end{cases}$$

Dispersion relation (plug normal modes into wave eqn)

$$-\omega^2 = \frac{\Delta x}{L}\,\frac{\Delta x}{C}(-k^2) \Rightarrow \omega = \underbrace{\sqrt{\frac{\Delta x}{L}\,\frac{\Delta x}{C}}}_{c_w = \text{ speed of wave}}\,k$$

Also note: the solutions for $V(x,t)$ and $I(x,t)$ are related:

recall eqn ② $V_n - V_{n+1} = L\,\frac{dI_n}{dt}$

$$\Rightarrow -\Delta x\,\frac{\partial V(x,t)}{\partial x} = L\,\frac{dI(x,t)}{dt}$$

plug in $V_0 e^{i(kx \pm \omega t)}$ and $I_0 e^{i(kx \pm \omega t)}$

$$\Rightarrow -\Delta x\,(ik)\,V_0 e^{i(kx \pm \omega t)} = L(\pm i\omega)\,I_0 e^{i(kx \pm \omega t)}$$

$$\Rightarrow \frac{V_0}{I_0} = \mp\frac{L}{\Delta x}\,\frac{\omega}{k} = \mp\frac{L}{\Delta x}\sqrt{\frac{\Delta x}{L}\,\frac{\Delta x}{C}} = \mp\sqrt{\frac{L}{C}}$$

Ohm's Law: $V = IR$

$\sqrt{\frac{L}{C}}$ is like resistance, but not quite

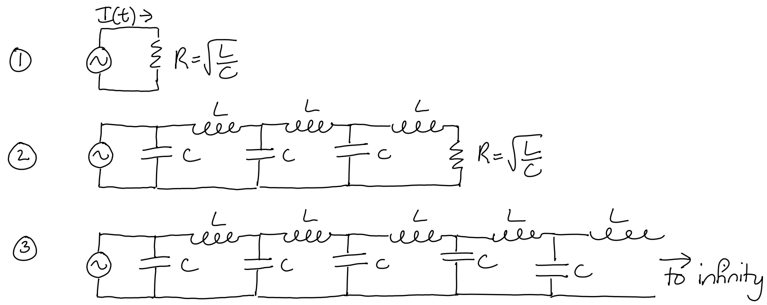
It is a "generalized resistance" called "impedance".

(R is a special form of impedance, called resistance.)

$\Rightarrow V(t) = Z\,I(t)$ where $Z = \sqrt{\frac{L}{C}}$

Implications of Impedance

From the point of view of the wave generator (power supply) the whole LC line looks identical to a simple resistor with $R=\sqrt{L/C}$.



From the point of view of the power supply, all 3 circuits are equivalent.

In case ② we say that "the line is terminated with a matching impedance."

Quiz

(a) Suppose you want to send a signal down a line with $L/\Delta x = 10^{-4}$ H/m and $C/\Delta x = 2.5\times10^{-11}$ F/m. What is the signal speed?

(b) What resistor should the recipient use to terminate the line, to avoid inadvertently reflecting the signal back?

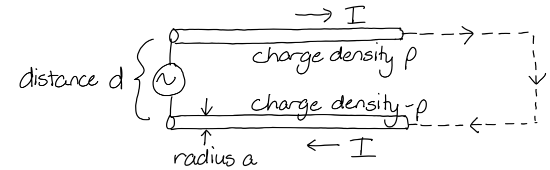
(c) What would the voltage pulse reflection look like for an open circuit ($R=\infty$)?

(d) What would the voltage pulse reflection look like for a short circuit ($R=0$)?

(e) Sneak preview: how much power do you think it takes to drive a wave $V_0\sin(\omega t)$ down the infinite LC line ($V_0 = 1V$)?

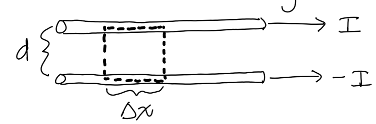
Example: 2 parallel wires

(e.g. phone line)



Inductance - defined by the magnetic field and current

$$\underset{\text{magnetic flux}}{\Phi} = L\,\underset{\text{current flowing}}{I}$$



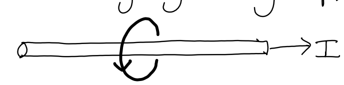
→ run current I and $-I$ on these wires

→ need to calculate resultant flux in $d\times\Delta x$ rectangle

→ need to compute B as a function of distance between wires

review: Ampere's Law

① Draw a closed loop around a current distribution (needs to be a high-symmetry loop, or you won't get useful info)



② Ampere's law states

$$\underbrace{\oint \vec{B}\cdot d\vec{l}}_{\text{line integral around loop}} = \mu_0 I_{enclosed}$$

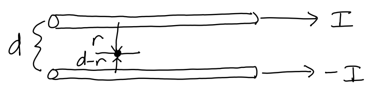
μ_0 = permeability of vacuum $= 4\pi\times10^{-7}$ H/m

③ If we choose the loop centered around a single symmetric current-carrying wire, we can do the line integral trivially:

$$2\pi r B = \mu_0 I \quad\Rightarrow\quad B = \frac{\mu_0 I}{2\pi r}$$

④ Check sign using right-hand rule.

Back to our inductance calculation:



At the marked point, a distance r from one wire & $d-r$ from other:

$$B(r) = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi(d-r)} \quad \text{(into page!)}$$

Integrate between wires to calculate the total flux:

$$\Phi = \int_0^{\Delta x} dx \int_a^{d-a} dr\, B(r) = \Delta x \int_a^{d-a} B(r)\,dr$$

$$= \Delta x \int_a^{d-a} \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} + \frac{1}{d-r}\right) dr$$

$$= \Delta x \frac{\mu_0 I}{2\pi}\left[\ln(r) - \ln(d-r)\right]\Big|_a^{d-a}$$

$$= \frac{\Delta x \mu_0 I}{2\pi}\left[\ln\left(\frac{d-a}{a}\right) - \ln\left(\frac{a}{d-a}\right)\right]$$

$$= \frac{\Delta x \mu_0 I}{\pi} \ln\left(\frac{d-a}{a}\right)$$

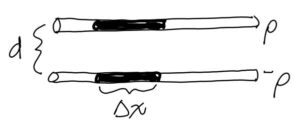
$$\approx \frac{\Delta x \mu_0 I}{\pi} \ln\left(\frac{d}{a}\right) \quad \text{in the limit } d \gg a$$

$$\Rightarrow L = \frac{\Phi}{I} = \frac{\frac{\Delta x \mu_0 I}{\pi}\ln\left(\frac{d}{a}\right)}{I} = \frac{\Delta x \mu_0}{\pi}\ln\left(\frac{d}{a}\right)$$

$$\Rightarrow \frac{L}{\Delta x} = \frac{\mu_0}{\pi}\ln\left(\frac{d}{a}\right)$$

Capacitance - defined by the voltage and charge

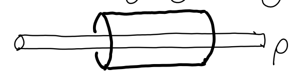
$$C = \frac{q}{V} \quad \begin{matrix} \leftarrow \text{charge stored} \\ \leftarrow \text{voltage applied} \end{matrix}$$



→ put charge density ρ and $-\rho$ on these wires
→ need to calculate resultant electrical potential between the wires
→ need to compute E as a function of distance between wires

review: Gauss' Law

① Draw a closed surface around a charge distribution (needs to be a high-symmetry surface, or you won't get useful info)



② Gauss' law states

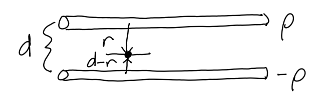
$$\underbrace{\oint \vec{E}\cdot d\vec{a}}_{\text{surface integral}} = \frac{q_{enclosed}}{\varepsilon_0}$$

$\leftarrow \varepsilon_0$ = permittivity of vacuum $= 8.85\times10^{-12}$ F/m

③ If we choose the surface centered around a single symmetric current-carrying wire, we can do the surface integral trivially. By symmetry, there can be no E-field pointed along the wire, so the surface integral over the cylinder ends vanishes.

$$2\pi r \Delta x E = \frac{\rho \Delta x}{\varepsilon_0} \Rightarrow E = \frac{\rho}{2\pi r \varepsilon_0}$$

Back to our capacitance calculation:



At the marked point, a distance r from one wire & $d-r$ from other:

$$E(r) = \frac{\rho}{2\pi r \varepsilon_0} + \frac{\rho}{2\pi(d-r)\varepsilon_0} \quad \text{(down, in plane of page!)}$$

Integrate between the wires to calculate the voltage:

$$V = \int_a^{d-a} E(r)\,dr$$

$$= \int_a^{d-a} \frac{\rho}{2\pi\varepsilon_0}\left(\frac{1}{r} + \frac{1}{d-r}\right) dr \quad \text{same integral we just did for inductance}$$

$$\approx \frac{\rho}{2\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)$$

$$C = \frac{q}{V} = \frac{\rho\Delta x}{\frac{\rho}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)} = \frac{\Delta x}{\pi\varepsilon_0 \ln\left(\frac{d}{a}\right)}$$

$$\Rightarrow \frac{C}{\Delta x} = \frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$

Quiz: what is the impedance of this transmission line?
what is the signal velocity in this transmission line?

Impedance

$$Z = \sqrt{\frac{L}{C}} = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln\left(\frac{d}{a}\right) = \frac{(377\,\Omega)}{\pi}\ln\left(\frac{d}{a}\right)$$

Wave velocity

$$c_w = \sqrt{\frac{\Delta x}{L}\frac{\Delta x}{C}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = \frac{1}{\sqrt{(8.85\times10^{-12}\,F/m)(4\pi\times10^{-7}\,H/m)}}$$

$$= 3\times10^8\,m/s = c = \text{speed of light!}$$

A pair of parallel wires in vacuum transmit electromagnetic waves at the speed of light!

What if the wires are immersed in some other medium?
- → just replace ε_0 and μ_0 with ε and μ of the medium.
- → wave velocity $c_w = \frac{1}{\sqrt{\varepsilon\mu}}$ is always smaller than c.

A parallel wire transmission in vacuum is the simplest possible
- → anything more complex has <u>at least</u> as much capacitance and inductance per unit length
- → this will make the wave velocity c_w smaller
- → wave velocity on an LC transmission line never exceeds c

Energy

How is energy carried by LC line?
Energy is in 2 forms:

① electrostatic energy in C: $E_C = \frac{1}{2}CV^2$

② magnetic energy in L: $E_L = \frac{1}{2}LI^2$

We also know that $V = ZI = \sqrt{\frac{L}{C}}\,I$

$$\Rightarrow E_C = \frac{1}{2}C\left(\sqrt{\frac{L}{C}}\,I\right)^2 = \frac{1}{2}LI^2 = E_L$$

⇒ energies in C and L are identical

Consider a forward-going wave $V(x,t) = V_0\cos(kx-\omega t)$
Total energy density is given by:

$$\frac{dE}{dx} = \frac{2E_C}{\Delta x} = \frac{CV^2}{\Delta x} = \frac{C}{\Delta x}V_0^2\cos^2(kx-\omega t)$$

Time average gives:

$$\left\langle\frac{dE}{dx}\right\rangle = \frac{1}{2}\frac{C}{\Delta x}V_0^2$$

Multiply by wave velocity c_w to get the transfer rate:

$$\langle P\rangle = \left\langle\frac{dE}{dt}\right\rangle = \frac{1}{2}c_w\frac{C}{\Delta x}V_0^2 = \frac{1}{2}\sqrt{\frac{\Delta x}{L}\frac{\Delta x}{C}}\,\frac{C}{\Delta x}V_0^2 = \frac{1}{2}\sqrt{\frac{C}{L}}V_0^2 = \frac{1}{2}\frac{V_0^2}{Z} = \frac{1}{2}V_0I_0$$

Ah-hah! this is exactly what we expected, because from the power supply's point of view it just has to supply voltage $V(t) = V_0\cos(\omega t)$ and current $I(t) = I_0\cos(\omega t)$

$$\langle P\rangle = \langle IV\rangle = \langle V_0I_0\cos^2(\omega t)\rangle = \frac{1}{2}V_0I_0$$

Where is the energy?

What does it mean that the energy is stored "in" L and C?

Energy in L is stored in the form of a magnetic field.
→ it's around, but not in the wire

$$\frac{dE_L}{dx} = \frac{1}{2}\frac{L}{\Delta x}I^2 = \frac{1}{2}\frac{\mu}{\pi}\ln\left(\frac{d}{a}\right)I^2$$

It depends on the μ of the material surrounding the wires.

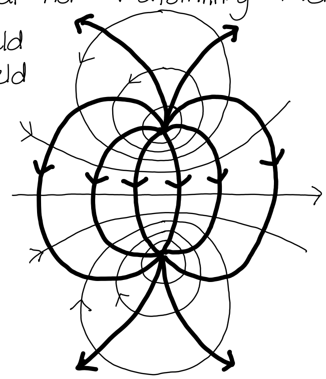
Energy in C is stored in the form of an electric field.
→ it's between, but not in the wire

$$\frac{dE_C}{dx} = \frac{1}{2}\frac{C}{\Delta x}V^2 = \frac{1}{2}\frac{\pi\varepsilon}{\ln(d/a)}V^2$$

It depends on the ε of the material surrounding the wires.

The energy is not carried by the physical wires, but in the space around them, as an electromagnetic field.
→ wires are guiding the waves but not transmitting them

— E-field
— B-field



parallel wire transmission:
E & B fields extend to infinity
→ leaky way of carrying energy
→ radiation loss = problem

Poynting vector

$$\vec{S} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\vec{B}}{\mu}$$

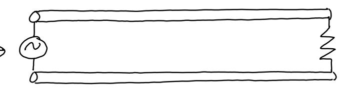
- perpendicular to both $\vec{E}$ and $\vec{B}$
  - ⇒ points in direction of wave transmission
- units $\vec{E}\,(V/m) \times \vec{H}\,(A/m)$
  - ⇒ Watts/m^2 = how much power is flowing per unit cross-sectional area

Momentum

Does a traveling wave on an LC transmission line carry momentum?

Yes! But we can't see any mass moving...
We can see the forces that are needed/produced at the ends.

To get some physical intuition for what these mysterious forces are, look back at the specific example of the parallel wire transmission.



to create waves, the power supply must run across the gap between the wires

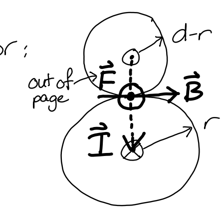
at the receiving end, resistor must also run across the gap between the wires

Current flows along the wires and through these components

→ Current flowing along the wires creates B-field between them
→ Current flowing through these components in the presence of the B-field feels a Lorentz force

Let's calculate the Lorentz force (for this specific example of parallel wire transmission line)

Look upstream from the terminating resistor:



Magnetic field between wires is:

$$B = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi(d-r)}$$

Lorentz force is: $\vec{F} = \vec{I}\times\vec{B}$

- $\vec{I}$ (through components) is perpendicular, from one wire to the other
- $\vec{B}$ is perpendicular, between wires

⇒ $\vec{F}$ is parallel to the direction of the wires
→ points backwards at the power supply
→ points forwards at the resistor

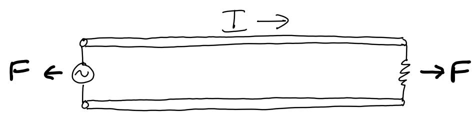
Integrate $\vec{I} \times \vec{B}$ to calculate the total force

$$\vec{F} = \int_a^{d-a} \vec{I}(r) \times \vec{B}(r)\,dr$$

$$= \frac{\mu_0 I^2}{2\pi} \int_a^{d-a} \left(\frac{1}{r} + \frac{1}{d-r} \right) dr$$

We did this integral before when computing inductance, etc. Recall: $\frac{L}{\Delta x} = \frac{\mu_0}{\pi} \ln\left(\frac{d}{a}\right)$

$$\approx \frac{\mu_0 I^2}{\pi} \ln\left(\frac{d}{a}\right) = \frac{L}{\Delta x} I^2$$



The waves transmit a force $F = \frac{L}{\Delta x} I^2$

Remember $F = \frac{dp}{dt}$ so this force is the rate of momentum transfer.

For normal mode waves, the driver must produce

$$V(t) = Z\,I(t) = V_0 \sin(\omega t)$$

Therefore, the force is: $F = \frac{L}{\Delta x} I_0^2 \sin^2(\omega t)$

Time average: $\langle F \rangle = \frac{1}{2} \frac{L}{\Delta x} I_0^2 = \frac{1}{2} \frac{L}{\Delta x} \frac{V_0}{Z} I_0$

$$= \frac{1}{2} \frac{L}{\Delta x} \left(\frac{\Delta x}{L} \frac{C}{\Delta x} \right)^{1/2} V_0 I_0 = \frac{1}{2} \left(\frac{L}{\Delta x} \frac{C}{\Delta x} \right)^{1/2} V_0 I_0$$

$\boxed{\langle F \rangle = \frac{1}{2} \frac{V_0 I_0}{C_w}}$ **momentum transfer**

Compare to:

$\boxed{\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} V_0 I_0}$ **energy transfer**

Summary

* LC transmission line wave equation:
$$\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\Delta x}{L} \frac{\Delta x}{C} \frac{\partial^2 V(x,t)}{\partial x^2} \qquad \text{(same for } q,\ I\text{)}$$
* impedance: $Z = \sqrt{L/C}$
* wave speed: $C_w = \sqrt{\frac{\Delta x}{L} \frac{\Delta x}{C}}$
* Ampere's Law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enclosed}$
* Gauss' Law: $\oint \vec{E} \cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_0}$
* speed of light: $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$
* energy: $\langle P \rangle = \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} V_0 I_0$
* momentum: $\langle F \rangle = \left\langle \frac{dp}{dt} \right\rangle = \frac{1}{2} \frac{V_0 I_0}{C_w}$

Next time: Electromagnetic waves in free space
Polarization

Quiz Answers

(a) $C_w = \sqrt{\frac{\Delta x}{L} \frac{\Delta x}{C}} = \sqrt{\frac{10^4\,m}{H} \cdot \frac{4\times10^{10}\,m}{F}} = \sqrt{4\times10^{14}}$ m/s $= 2\times10^7$ m/s

(b) $R = \sqrt{\frac{L}{C}} = \sqrt{\frac{10^{-4}\,H/m}{2.5\times10^{-11}\,F/m}} = \sqrt{4\times10^6}\,\Omega = 2\,k\Omega$

(c) full reflection, right-side-up

(d) full reflection, up-side-down

(e) $P = IV = \frac{V_0^2 \sin^2(\omega t)}{\sqrt{L/C}} \Rightarrow \langle P \rangle = \frac{1}{2} \frac{V_0^2}{\sqrt{L/C}} = \frac{1}{2} \frac{(1V)^2}{(2\,k\Omega)} = 2.5\times10^{-4}$ W

Q: Compared to our old formula for the power transfer by a mechanical wave, $\langle P \rangle = \frac{1}{2} Z \omega^2 \xi_0^2$, where is the frequency dependence?

A: It's hiding inside V_0: $V = IR$ and $I = dq/dt$, so I gets a factor of ω from the time derivative.