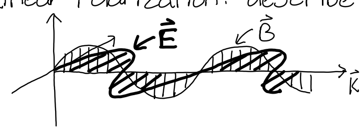
Physics 15c (Hoffman)
Lecture #16
Thurs, Oct 28, 2010

Electromagnetic Waves in Matter: Reflection & Refraction

Last time:

* Reviewed Maxwell's equations and derived wave equations for electric and magnetic fields
* $\vec{E}$ and $\vec{B}$ are transverse waves and perpendicular to each other.
* Intensity (power/area) of EM waves: $I = \frac{1}{2} c\varepsilon_0 E_0^2$
* Polarization = direction of $\vec{E}$ for EM waves
* Linear Polarization: describe any $\vec{E}$ direction using this basis:



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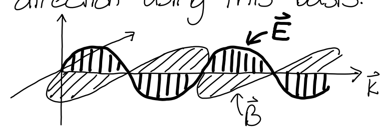
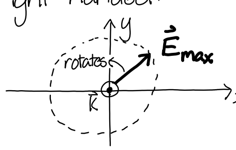
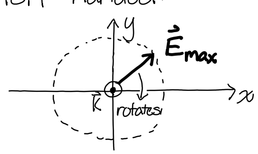


* Malus' law: $I_{out} = I_{in} \cos^2\theta$ (θ = angle between 2 polarizers)
* Circular Polarization: direction of max $\vec{E}$ rotates around $\vec{k}$-axis

right-handed:



left-handed:



Goals for today: EM waves in matter

* Conductors: skin depth
* Insulators: reflection & transmission @ normal incidence
* Huygen's principle
* Snell's law
* total internal reflection
* Fresnel coefficients: reflection & transmission @ non-normal incidence
* Brewster's angle

Conductors

Electric field in a conductor causes a current $\vec{J} = \sigma\vec{E}$
(where σ = conductivity)

⇒ we need the full form of Maxwell's equations

Gauss' Law: ① $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

No name: ③ $\vec{\nabla}\cdot\vec{B} = 0$

Faraday's Law: ② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

④ $\vec{\nabla}\times\vec{B} = \underbrace{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{Maxwell's correction}} + \underbrace{\mu_0\vec{J}}_{\text{Ampere's Law}}$

In Maxwell's equations above, ρ and $\vec{J}$ are the total charge and current density. It's more convenient to talk about the free ρ and free $\vec{J}$; in order to do this, we replace $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$.

[Note: these are valid in any linear medium, where $\vec{D} = \varepsilon\vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$]

① $\vec{\nabla}\cdot\vec{E} = \frac{\rho_{free}}{\varepsilon}$ ③ $\vec{\nabla}\cdot\vec{B} = 0$

② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ ④ $\vec{\nabla}\times\vec{B} = \varepsilon\mu\frac{\partial\vec{E}}{\partial t} + \mu\vec{J}_{free}$

$\vec{J}_{free} = \sigma\vec{E}$

Do our same old math tricks to derive wave equations in conductors:

$$\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} + \underbrace{\mu\sigma\frac{\partial\vec{E}}{\partial t}} \qquad \nabla^2\vec{B} = \varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2} + \underbrace{\mu\sigma\frac{\partial\vec{B}}{\partial t}}$$

new term looks like damping

We can still find plane waves $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ and $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$
but when we plug into the wave equation, we get a different dispersion relation:

$$\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \Rightarrow k^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$$

⇒ k is imaginary!

Redefine k to split it into real and imaginary parts:

$$(k + i\mathcal{K})^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$$

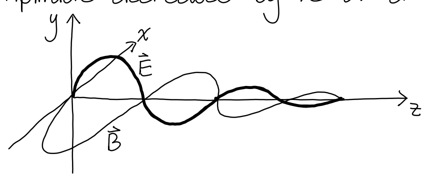
$\mathcal{K}$ ← kappa

⇒ waves moving along +z axis look like

$$\vec{E} = \vec{E}_0 e^{-\mathcal{K}z} e^{i(kz-\omega t)} \qquad \vec{B} = \vec{B}_0 e^{-\mathcal{K}z} e^{i(kz-\omega t)}$$

→ EM waves in a conductor decay exponentially

→ amplitude decreases by $1/e$ at $d = 1/\mathcal{K}$ ≡ "skin depth"



Typical metal:
$\sigma \sim 10^7$ A/(V·m)
ε ~ very large

Solve for k and $\mathcal{K}$: $(k+i\mathcal{K})^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$

$$k^2 - \mathcal{K}^2 = \varepsilon\mu\omega^2 \,;\; 2k\mathcal{K} = \mu\sigma\omega$$

$$k^2 - \left(\frac{\mu\sigma\omega}{2k}\right)^2 = \varepsilon\mu\omega^2 \qquad \left(\frac{\mu\sigma\omega}{2\mathcal{K}}\right)^2 - \mathcal{K}^2 = \varepsilon\mu\omega^2$$

$$k^4 - \varepsilon\mu\omega^2 k^2 - \tfrac{1}{4}\mu^2\sigma^2\omega^2 = 0 \qquad \mathcal{K}^4 + \varepsilon\mu\omega^2\mathcal{K}^2 - \tfrac{1}{4}\mu^2\sigma^2\omega^2 = 0$$

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)^{1/2} \qquad \mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)^{1/2}$$

When $\omega \to$ small, then $\frac{\varepsilon\omega}{\sigma} \ll 1$

$$\mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{\sigma}{\varepsilon\omega}\left(\sqrt{1+\left(\frac{\varepsilon\omega}{\sigma}\right)^2} - \frac{\varepsilon\omega}{\sigma}\right)\right]^{1/2}$$

$$= \sqrt{\frac{\mu\sigma\omega}{2}}\Big[1 + \underbrace{\tfrac{1}{2}\left(\tfrac{\varepsilon\omega}{\sigma}\right)^2 - \tfrac{\varepsilon\omega}{\sigma}}_{\text{small}}\Big]^{1/2} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$$

→ $\mathcal{K}$ grows with frequency

→ at low frequency, $\mathcal{K} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$

→ skin depth gets shorter as ω↑

→ high f is stopped by thin metal

Insulators

Insulators are even easier than conductors because we have no free charge or current. So Maxwell's eqns reduce to:

① $\vec{\nabla}\cdot\vec{E} = 0$ ③ $\vec{\nabla}\cdot\vec{B} = 0$

② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ ④ $\vec{\nabla}\times\vec{B} = \varepsilon\mu\frac{\partial\vec{E}}{\partial t}$

So we are exactly back to the vacuum situation except we have replaced $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$.

Note: ε and μ may be ω-dependent in a dispersive medium. We've also just seen (e.g. 1/4-wave plate) that they can be (although not usually) direction dependent.

→ Speed of light becomes

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \to \frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu}} \quad \text{where } n = \text{"index of refraction"} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

→ Plane wave solutions look the same

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \text{ and } \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad \left(\text{with dispersion } \omega = \frac{1}{\sqrt{\varepsilon\mu}}|\vec{k}|\right)$$

→ Power density is

$$\langle|\vec{S}|\rangle = \frac{E_0^2}{2z} \quad \text{where } z = \sqrt{\frac{\mu}{\varepsilon}} = \text{impedance of material}$$

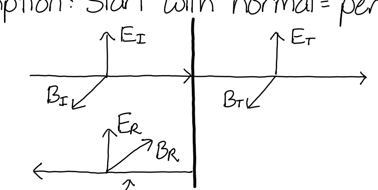
Reflection and Transmission

What happens when EM waves cross the boundary between 2 insulators? (e.g. light from air hits a glass window)

Simplifying assumption: start with normal = perpendicular incidence

Define:

$$\begin{cases} \vec{E}_1 = \vec{E}_I + \vec{E}_R \\ \vec{B}_1 = \vec{B}_I + \vec{B}_R \end{cases} \qquad \begin{cases} \vec{E}_2 = \vec{E}_T \\ \vec{B}_2 = \vec{B}_T \end{cases}$$



Note that Maxwell's equations told us that $\vec{k}\times\vec{E} = \omega\vec{B}$ so when $\vec{k}$ flips direction upon reflection, if $\vec{E}$ stays the same, then $\vec{B}$ must flip too

At the boundary:

① $E_1^{\parallel} = E_2^{\parallel}$ (charge accumulation at boundary <u>can't</u> change in-plane electric field)

$$\Rightarrow \vec{E}_I + \vec{E}_R = \vec{E}_T$$

② $H_1^{\parallel} = H_2^{\parallel}$ (current flow at boundary <u>can</u> change in-plane magnetic field → need to use $\vec{H}$)

$$\Rightarrow \vec{H}_I + \vec{H}_R = \vec{H}_T$$

Recall that $|\vec{H}| = \frac{|\vec{B}|}{\mu} = \frac{|\vec{E}|}{c\mu} = \frac{|\vec{E}|}{Z}$

So the boundary conditions reduce to

① $E_I + E_R = E_T$

② $\frac{E_I}{Z_1} - \frac{E_R}{Z_1} = \frac{E_T}{Z_2}$

note the sign, because B flips direction

Solve these equations ① and ② ⇒ find reflection & transmission:

$$E_R = \frac{Z_2 - Z_1}{Z_1 + Z_2} E_I \qquad H_R = \frac{Z_1 - Z_2}{Z_1 + Z_2} H_I \qquad S_R = -\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 S_I$$

$$E_T = \frac{2Z_2}{Z_1 + Z_2} E_I \qquad H_T = \frac{2Z_1}{Z_1 + Z_2} H_I \qquad S_T = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} S_I$$

Note: index of refraction n is defined by $c_w = \frac{c}{n}$, i.e. $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

but for most insulators, $\mu \approx \mu_0$

$\Rightarrow$ approximate $n \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}}$ $\quad \Rightarrow Z = \sqrt{\frac{\mu}{\varepsilon}} \approx \sqrt{\frac{\mu_0}{\varepsilon}} = \frac{Z_0}{n}$

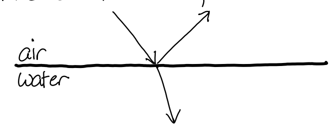
Many textbooks use this approximation, sometimes without even telling you that they're using it, b/c people are more familiar with n than Z.

e.g. reflectivity is: $R = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \approx \left(\frac{1/n_1 - 1/n_2}{1/n_1 + 1/n_2}\right)^2 = \left(\frac{n_2 - n_1}{n_1 + n_2}\right)^2$

Quiz: How much power is reflected back when light hits glass?
(Take $Z_{air} \approx Z_{vacuum} = 377\,\Omega$; $Z_{glass} \approx 250\,\Omega$.)

Reflection & Transmission @ Non-Normal Incidence: Angles

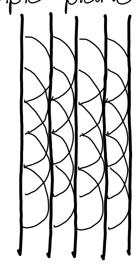
What if light hits an insulator, but not perpendicularly?



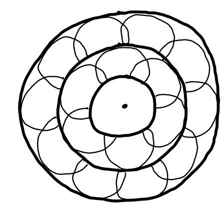
Huygen's principle

→ draw circles to construct successive wavefronts
→ each circle has $r = \lambda$ and is centered on the previous wavefront
→ draw a common tangent of all circles
→ get new wavefront

example: plane wave



example: circular wave (point source)

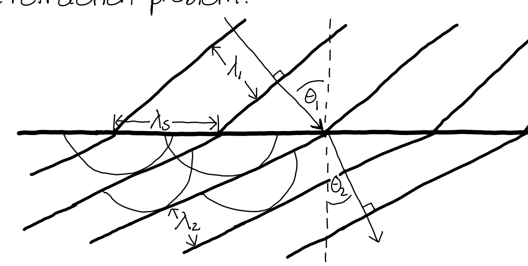


Snell's Law

Apply Huygen's principle to the refraction problem:

→ wavelength λ changes at the boundary

$\lambda_1 = \frac{c}{n_1 f}$ $\qquad \lambda_2 = \frac{c}{n_2 f}$



→ consider wavelength along the surface

$$\lambda_s = \frac{\lambda_1}{\sin\theta_1} = \frac{\lambda_2}{\sin\theta_2}$$

$$\Rightarrow \boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1}}$$ **Snell's Law** = law of refraction

Total internal reflection

Direction of refraction "bend" depends on n_1/n_2.

fast → slow $\Rightarrow n_1 < n_2 \Rightarrow \theta_1 > \theta_2 \Rightarrow$ bend "down" (like air & water)

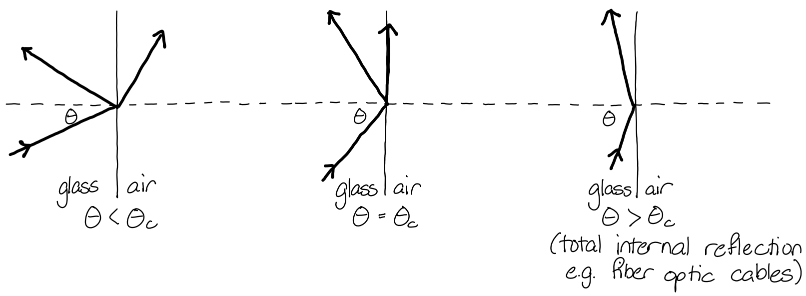
slow → fast $\Rightarrow n_1 > n_2 \Rightarrow \theta_2 > \theta_1 \Rightarrow$ bend up

But for slow → fast transition, something weird can happen:

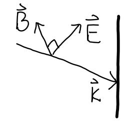
$\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 \leq \frac{n_2}{n_1} \leq 1$ [because $\sin(\text{anything}) \leq 1$]

$\Rightarrow \boxed{\theta_1 \leq \sin^{-1}\left(\frac{n_2}{n_1}\right)}$ **"critical angle"**

example: glass $(n=1.5)$ → air $(n \approx 1)$ $\Rightarrow \theta_c = \sin^{-1}\left(\frac{2}{3}\right) = 42°$



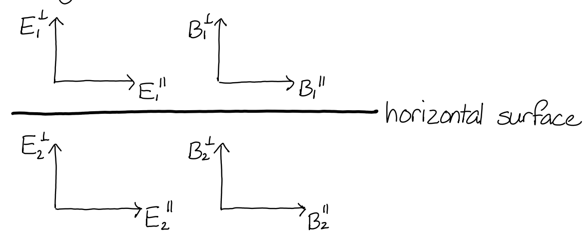
Reflection & Transmission @ Non-Normal Incidence: Amplitudes



Now we need to consider components of $\vec{E}$ and $\vec{B}$ both parallel to and perpendicular to the surface.

→ We need to think carefully about polarization too!

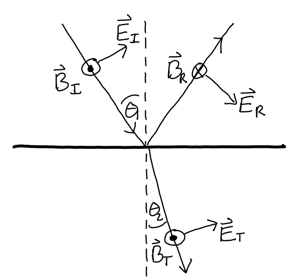
Boundary conditions



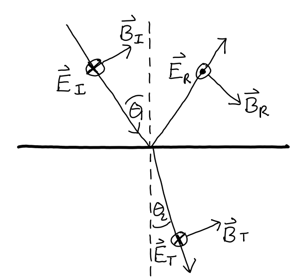
$$\begin{cases} E_1^{\parallel} = E_2^{\parallel} & \\ D_1^{\perp} = D_2^{\perp} \Rightarrow \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} & \text{(takes into account surface charge)} \\ H_1^{\parallel} = H_2^{\parallel} \Rightarrow (1/\mu_1) B_1^{\parallel} = (1/\mu_2) B_2^{\parallel} & \text{(takes into account surface current)} \\ B_1^{\perp} = B_2^{\perp} & \end{cases}$$

Must consider 2 different polarizations:

"Vertical"



"Horizontal"



Vertical polarization

① $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \Rightarrow (E_I + E_R)\cos\theta_1 = E_T\cos\theta_2$

② $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \Rightarrow \varepsilon_1(E_I - E_R)\sin\theta_1 = \varepsilon_2 E_T \sin\theta_2$

③ $H_1^{\parallel} = H_2^{\parallel} \Rightarrow H_I - H_R = H_T$

$B_1^{\perp} = B_2^{\perp} \Rightarrow$ nothing $(0=0)$

Also use connection between $\vec{E}$ and $\vec{H}$:

④ $H_I = \dfrac{E_I}{Z_1}$ ⑤ $H_R = \dfrac{E_R}{Z_1}$ ⑥ $H_T = \dfrac{E_T}{Z_2}$ $\left(Z = \sqrt{\dfrac{\mu}{\varepsilon}}\right)$

6 eqns; unknowns are: E_T, E_R, H_T, H_R, θ_2

Solve:

eliminate H's using ④, ⑤, ⑥

③ $\Rightarrow \dfrac{E_I - E_R}{Z_1} = \dfrac{E_T}{Z_2}$

combine with ②

$\Rightarrow Z_1\varepsilon_1\sin\theta_1 = Z_2\varepsilon_2\sin\theta_2$

recall: $Z\varepsilon = \sqrt{\varepsilon\mu} = 1/c_{wave} = n/c$

$\Bigg\}$ $n_1\sin\theta_1 = n_2\sin\theta_2$ Snell's law!

We are left with 2 eqns:

① $(E_I + E_R)\cos\theta_1 = E_T\cos\theta_2$

② $\varepsilon_1(E_I - E_R)\sin\theta_1 = \varepsilon_2 E_T\sin\theta_2$

Brute force...

① $E_R = \dfrac{E_T\cos\theta_2 - E_I\cos\theta_1}{\cos\theta_1}$

② $\varepsilon_1 E_I\sin\theta_1 - \dfrac{\varepsilon_1 E_T\cos\theta_2\sin\theta_1}{\cos\theta_1} + \varepsilon_1 E_I\sin\theta_1 = \varepsilon_2 E_T\sin\theta_2$

$$E_T = \frac{2\varepsilon_1\sin\theta_1\cos\theta_1}{\varepsilon_2\sin\theta_2\cos\theta_1 + \varepsilon_1\cos\theta_2\sin\theta_1}E_I$$

$$E_R = \frac{E_T\cos\theta_2}{\cos\theta_1} - E_I = \left[\frac{2\varepsilon_1\sin\theta_1\cos\theta_2}{\varepsilon_2\sin\theta_2\cos\theta_1 + \varepsilon_1\cos\theta_2\sin\theta_1} - 1\right]E_I$$

$$E_R = \frac{\varepsilon_1\sin\theta_1\cos\theta_2 - \varepsilon_2\cos\theta_1\sin\theta_2}{\varepsilon_1\sin\theta_1\cos\theta_2 + \varepsilon_2\cos\theta_1\sin\theta_2}E_I$$

$$E_T = \frac{2\varepsilon_1\sin\theta_1\cos\theta_1}{\varepsilon_1\sin\theta_1\cos\theta_2 + \varepsilon_2\cos\theta_1\sin\theta_2}E_I$$

Simplify by introducing:

$$\alpha = \frac{\cos\theta_2}{\cos\theta_1} \quad \text{and} \quad \beta = \frac{\varepsilon_2\sin\theta_2}{\varepsilon_1\sin\theta_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{Z_1}{Z_2}$$

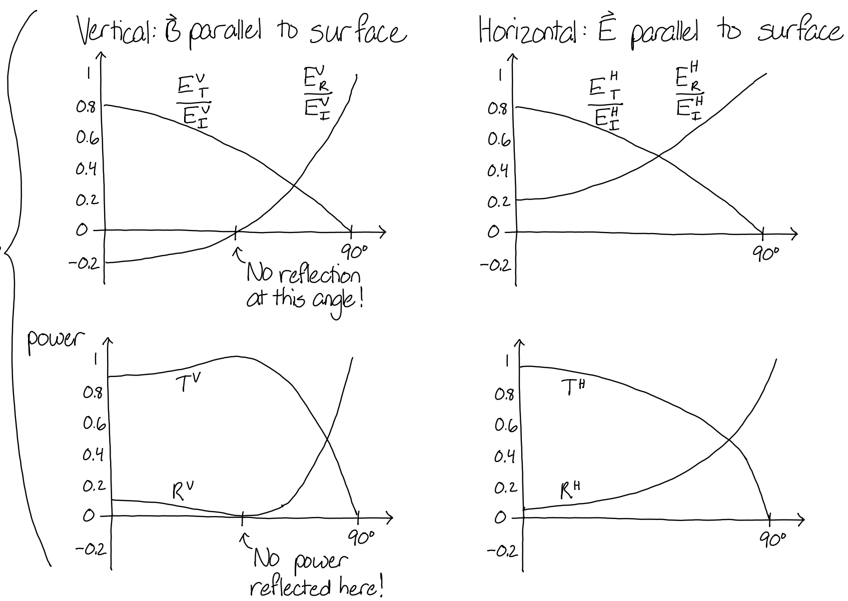
(Note: α is a function of the angle of incident light but β is just a constant for given 2 materials)

Vertical Polarization $\Rightarrow E_R = \dfrac{\alpha - \beta}{\alpha + \beta}E_I$ $\quad E_T = \dfrac{2}{\alpha + \beta}E_I$ $\Big\}$ Fresnel coefficients!

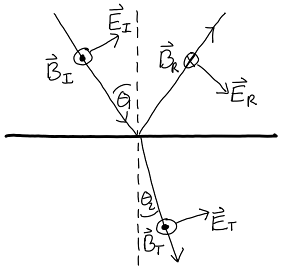
Could also solve horizontal polarization (HW):

Horizontal Polarization $\Rightarrow E_R = \dfrac{\alpha\beta - 1}{\alpha\beta + 1}E_I$ $\quad E_T = \dfrac{2}{\alpha\beta + 1}E_I$

Plotted for air→glass ($\beta = 1.5$)



Brewster's Angle



$E_R^V = \frac{\alpha - \beta}{\alpha+\beta} E_I^V$ where $\alpha = \frac{\cos\theta_2}{\cos\theta_1}$ and $\beta = \frac{Z_1}{Z_2}$

but when $\alpha = \beta$, light does not reflect!

$\frac{\cos\theta_2}{\cos\theta_B} = \beta$

Brewster's angle of no reflection

Use Snell's law: $n_2 \sin\theta_2 = n_1 \sin\theta_1$

$\Rightarrow \cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B}$

$1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B = \frac{Z_1^2}{Z_2^2}\cos^2\theta_B$

(algebra & approximate $\mu_1 \approx \mu_2$

$\tan\theta_B = \frac{n_2}{n_1}$

$\Rightarrow$ if light of all polarizations is incident on a surface at θ_B then only the horizontal component will reflect!

In fact, look at reflection curves for vertical & horizontal
$\rightarrow$ unless you are looking at a surface with near-normal angle, $R^H \gg R^V$ over a wide range of glancing angles

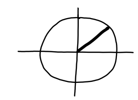
$\Rightarrow$ reflected light is primarily horizontally polarized

$\Rightarrow$ sunglasses are vertical polarizers

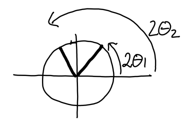
Is there any deeper significance to Brewster's angle?
Look at the condition $\alpha = \beta$ again:

$\alpha = \frac{\cos\theta_2}{\cos\theta_1}$ $\quad \beta = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} \approx \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \approx \frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}$

$\Rightarrow \frac{\cos\theta_2}{\cos\theta_1} = \frac{\sin\theta_1}{\sin\theta_2} \Rightarrow \sin(2\theta_1) = \sin(2\theta_2)$



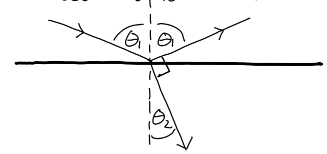
or



$\theta_1 = \theta_2$ $\downarrow$ not generally true

$\theta_1 + \theta_2 = 90°$ $\downarrow$ this must be true!

What does this look like?



At Brewster's angle, $\theta_1 + \theta_2 = 90°$

$\Rightarrow$ reflected and refracted light are perpendicular to each other

Is this a coincidence? Stay tuned...

Summary

* Conductors: skin depth, $1/d = \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right)^{1/2}$
* Insulators: replace $\varepsilon_0 \rightarrow \varepsilon, \mu_0 \rightarrow \mu \Rightarrow c_W = \frac{c}{n}$ where $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$
* Huygen's principle: draw successive wavefronts using point sources on previous wavefront
* Snell's law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$
* Total internal reflection: $\theta_C = \sin^{-1}(n_1/n_2)$ [where $n_2 > n_1$]
* Reflection & refraction @ non-normal incidence $\rightarrow$ Fresnel coeffs
  Vertical polarization ($\vec{B}$ parallel to surface) $\quad E_R = \frac{\alpha-\beta}{\alpha+\beta} \quad E_T = \frac{2}{\alpha+\beta}$
  [where $\alpha = \cos\theta_2/\cos\theta_1$ and $\beta = Z_1/Z_2$]
* Brewster's angle: $\tan\theta_B = n_2/n_1$

Next time: electromagnetic radiation (from moving charges)