Reading: H&L 1.6
or
Georgi 2.1
or
Morin 1.2

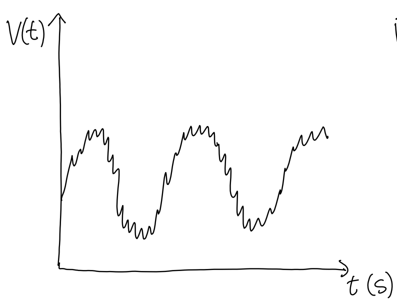
Physics 15c (Hoffman)
Lecture #2
Tues, Sept 7, 2010

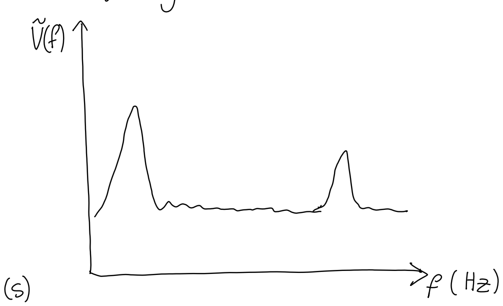
Math: Complex Numbers & Differential Eqns

Physics: Damped Oscillator

Last time:

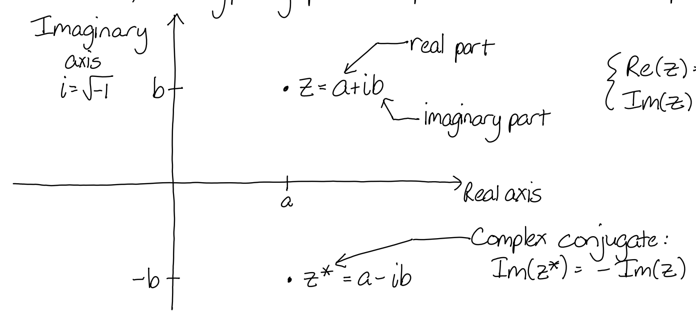
* mass-on-spring: found 2 solutions which were
  "linearly independent" = can't sum to zero unless coefficients are all zero
  "complete" = every solution can be written as a linear combo of these 2 (no 3rd soln needed)
* conservation of energy → "energy sloshing" between kinetic & potential
* why almost everything in the world that you care about is really a simple harmonic oscillator
  → Taylor expand an arbitrary conservative potential to find **linear** behavior near equilibrium
* Fourier transforms: **linearity** of solutions to oscillator eqns (and wave eqns, as we will find later) allows us to write the solution as a sum of oscillations at different frequencies
  → allows us to view the same information in the time domain or frequency domain





Complex #'s

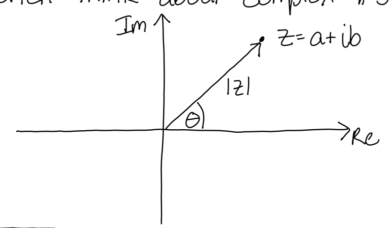
To visualize, we typically plot complex #'s in the "complex plane":



$\mathrm{Re}(z) = a$
$\mathrm{Im}(z) = b$

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \qquad \mathrm{Im}(z) = \frac{z - z^*}{2}$$

We also often think about complex #'s in polar coordinates:



$|z| = \sqrt{a^2+b^2}$ = distance from origin to z = "absolute value" or "magnitude"

$\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ = "argument" or "phase"

z may be expressed as:

$$z = |z|(\cos\theta + i\sin\theta) = \underbrace{|z|e^{i\theta}}$$

Whoa! Let's look at this in more detail...

Taylor expand e^x:

$$e^x = 1 + x\underbrace{\frac{d}{dx}(e^x)\Big|_0}_{e^x|_0=1} + \frac{1}{2!}x^2\underbrace{\frac{d^2}{dx^2}(e^x)\Big|_0}_{e^x|_0=1} + \frac{1}{3!}x^3\underbrace{\frac{d^3}{dx^3}(e^x)\Big|_0}_{e^x|_0=1} + \dots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

Likewise, we could Taylor expand e^{ix}:

$$e^{ix} = 1 + (ix) + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \dots$$

$$= 1 + ix - \frac{1}{2!}x^2 - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 + \frac{i}{5!}x^5 + \dots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$+ i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)$$

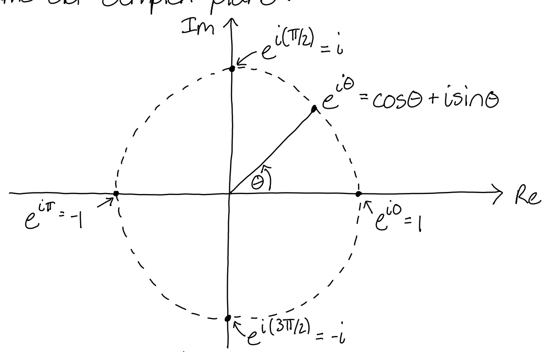
Likewise, we could Taylor expand $\sin\theta$ and $\cos\theta$ (HW)

→ can prove by comparison that

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

We often talk about the "unit circle"

same old complex plane:



Magnitude of $e^{i\theta}$ is always 1.

Linear n^{th} order differential equations:

First, a brief review of complex algebra:

$$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_2 z^2 + a_1 z + a_0 = 0$$

n^{th} order polynomial equation always has exactly n complex roots.

This is called the "Fundamental Theorem of Algebra"

proof: somewhat technical, but the basic idea is first we show that we can factor out one root:

$$\underbrace{(z - b_1)}_{1^{st}\text{ root}}\underbrace{(a_n' z^{n-1} + a_{n-1}' z^{n-2} + \dots + a_1')}_{\text{remaining }(n-1)^{th}\text{ order polynomial}} = 0$$

then we proceed by induction.

Note:

* Some roots might be identical, e.g. $x^2 - 2x + 1 = 0$
  2 roots are both $+1$
* For polynomial eqns with all real coeffs (i.e. $\{a_n, a_{n-1}, \dots a_1, a_0\} \in \mathbb{R}$):
  if $\alpha = a + ib$ is a root, then complex conjugate $\alpha^* = a - ib$ is also a root
  e.g. $x^2 + 1 = 0 \rightarrow$ 2 roots are $\pm i$

Note: since all of our equations of motion in this course do describe the real world, they will always have all real coefficients.

What does this have to do with differential equations?

n^{th} order "homogenous" linear differential equations:

"homogenous": means no terms that don't depend on x at all

linear: means no term has a power of x greater than one

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_1\frac{dx}{dt} + a_0 x = 0$$

→ can factor, just like a polynomial:

$$a_n\left(\frac{d}{dt} - \alpha_1\right)\left(\frac{d}{dt} - \alpha_2\right)\cdots\left(\frac{d}{dt} - \alpha_n\right)x = 0$$

→ we're left with n separate 1^{st} order linear differential equations, which are easy to solve

Single 1st order diff eqn:

$$\left(\frac{d}{dt} - \alpha_1\right)x = 0$$

$$\frac{dx}{dt} = \alpha_1 x$$

→ solve by inspection: what function $x(t)$ has a derivative that's just a constant times itself?

answer: exponential

→ plug in trial solution $x(t) = A_1 e^{\alpha_1 t}$

$$A_1\alpha_1 e^{\alpha_1 t} = \alpha_1 A_1 e^{\alpha_1 t}$$ ✓ OK, it works

Now for our total n^{th} order diff. eqn, the full soln is:

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \ldots + A_n e^{\alpha_n t}$$

where A_i's and α_i's are in general complex numbers

Degeneracy:

* What if we have a double root, i.e. $\alpha_i = \alpha_j$? ← this is called a "degeneracy"

short answer: if $\alpha_i = \alpha_j$ then

$$A_i e^{\alpha_i t} + A_j e^{\alpha_j t} = (A_i + A_j) e^{\alpha_i t}$$

= only one linearly independent solution
⇒ can't satisfy 2 initial conditions
→ need another root
→ take 2nd root to be $A_j t e^{\alpha_i t}$
→ see details in HW problem

Ok, for non-degenerate cases, the point is that the solution looks like $Ae^{\alpha t}$ for complex α.

<u>Example</u>

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{(familiar mass + spring)}$$

plug in our trial solution $Ae^{\alpha t}$

$$\rightarrow \alpha^2 A e^{\alpha t} + \frac{k}{m} A e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 = -k/m$$

$$\Rightarrow \alpha = \pm i\sqrt{k/m} = \pm i\omega_0$$

$$x(t) = e^{\pm i\omega_0 t}$$

→ Good, complex conjugate is soln too, as expected.

Hold on! What in the world have we just done?!?!

how can a mass be at position $e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$?
just where exactly is the mass at time $t = \pi/2$?
at position i?

No, of course not, the mass must be at a real position at all times.

The point is that we must look at the whole solution:

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$ (where A and B may be, in general, imaginary)

→ solve for A and B using x_0, v_0
→ must come out with A & B s.t. whole soln is ALWAYS real

e.g. $x_0 = x_{max}$, $v_0 = 0$

$$x(t) = A\cos(\omega_0 t) + iA\sin(\omega_0 t) + B\cos(\omega_0 t) - iB\sin(\omega_0 t)$$
$$= (A+B)\cos(\omega_0 t) + i(A-B)\sin(\omega_0 t)$$
$$v(t) = -\omega_0(A+B)\cos(\omega_0 t) + i\omega_0(A-B)\cos(\omega_0 t)$$

$$\Rightarrow \begin{cases} A+B = x_{max} \\ A-B = 0 \end{cases}$$

$$\Rightarrow A = B = x_{max}/2$$

More generally, we can derive the conditions on A & B that must hold for any initial conditions, such that $x(t)$ is always real.

$$x(t) = \underbrace{(A+B)}_{\text{need this real}}\cos(\omega_0 t) + i\underbrace{(A-B)}_{\text{need this imaginary}}\sin(\omega_0 t)$$

$\Rightarrow Im(A) = -Im(B)$ $\quad\quad$ $\Rightarrow Re(A) = Re(B)$

$\Rightarrow A = B^*$ (A & B are complex conjugates)

Another way to think of this is in terms of "changing the basis" in our solution space.

We can write our solution as

① $x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

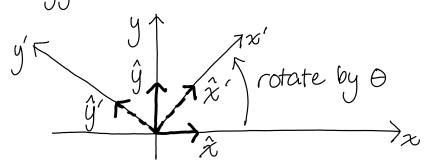
or

② $x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$ where $\begin{cases} C = A+B \\ D = i(A-B) \end{cases}$

One set of basis vectors is $\{e^{i\omega_0 t}, e^{-i\omega_0 t}\}$

and a different set of basis vectors spanning the same solution space is $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$.

Geometrical analogy: consider $\mathbb{R}^2$



- $\hat{x}$ and $\hat{y}$ are basis vectors which span the whole space $\mathbb{R}^2$
  → means every vector $\vec{r}$ in $\mathbb{R}^2$ can be written as $\vec{r} = a\hat{x} + b\hat{y}$
- but we could equally well choose $\hat{x}'$ and $\hat{y}'$ as an alternative set of basis vectors to span $\mathbb{R}^2$
  → the same $\vec{r}$ could also be written as $\vec{r} = c\hat{x}' + d\hat{y}'$ for a different pair of constants c and d

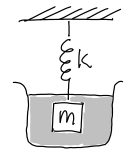
Why did we take this complicated excursion from the physical & intuitive $A\cos(\omega t) + B\sin(\omega t)$ to complex $Ae^{i\omega t} + Be^{-i\omega t}$?

→ to get a more general method to solve linear n^{th} order diff. eqns, b/c we won't always be so lucky & insightful as to guess 2 simple functions sine & cosine as we did for the mass & spring.

It's generally more convenient to use the $\{e^{i\omega t}, e^{-i\omega t}\}$ basis when first solving the diff. eqn, but to switch to the $\{\cos(\omega t), \sin(\omega t)\}$ basis when finding initial conditions.

Damped oscillator

So now we have this fancy way to solve linear differential eqns, but we need an example.



mass-on-a-spring with a twist: damping

N's 2^{nd} law: $\sum F = ma$

force from spring: $F_{spr} = -kx$

force from damping: $F_{damp} = -bv$

damping constant (b), proportional to velocity (v)

$$-kx - bv = ma$$
$$ma + bv + kx = 0$$
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \quad \Big\} \; 2^{nd} \text{ order linear homogenous differential eqn.}$$

Plug in our trial solution: $x(t) = Ae^{\alpha t}$

$$\alpha^2 Ae^{\alpha t} + \frac{b}{m}\alpha Ae^{\alpha t} + \frac{k}{m}Ae^{\alpha t} = 0$$
$$\alpha^2 + \frac{b}{m}\alpha + \underbrace{\frac{k}{m}}_{\omega_0^2} = 0$$

solve for α:

$$\alpha = \frac{-b/m \pm \sqrt{(b/m)^2 - 4\omega_0^2}}{2}$$
$$= -\frac{b}{2m} \pm \underbrace{\sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}}$$

this part could be real or imaginary, or even zero!
→ need to examine several cases

① **Small damping**: $b <$ what?

need something of the same dimensionality as b to compare to, in order to decide when b is small

→ look at square root:

if $\frac{b}{2m} < \omega_0$

$\frac{kg}{s} \cdot \frac{1}{kg}$ $\frac{1}{s}$

then square root is imaginary!

So in the case of small damping when $\frac{b}{2m} < \omega_0$

$$\alpha = -\frac{b}{2m} \pm i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$= -\gamma \pm i\omega \quad \text{where } \gamma \equiv \frac{b}{2m} \quad \text{and } \omega \equiv \sqrt{\omega_0^2 - \gamma^2}$$

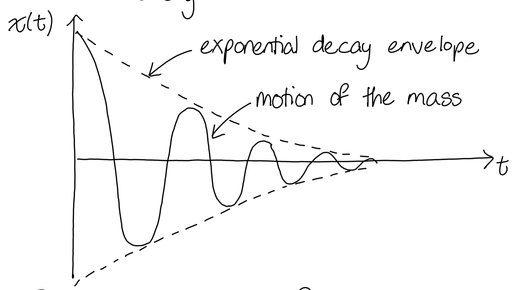
Therefore, the general solution is:

$$x(t) = A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t}$$

Note: these are subscripts on A and α. We're not subtracting t from α.

$$= A_+ e^{-\gamma t} e^{i\omega t} + A_- e^{-\gamma t} e^{-i\omega t}$$

$$= \underbrace{e^{-\gamma t}}_{\text{exponential decay}} \underbrace{\left(A_+ e^{i\omega t} + A_- e^{-i\omega t}\right)}_{\text{oscillation}}$$



The technical name for this case is "**underdamped**"

② **Large damping**: $\frac{b}{2m} > \omega_0$

now the square root is real, so we write

$$\alpha = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}$$

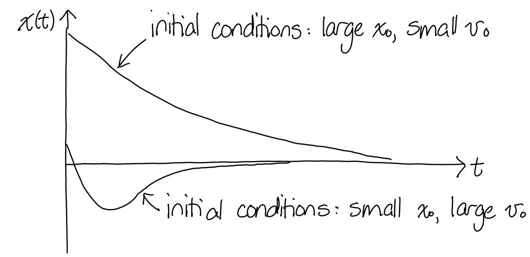
$$= -\gamma \pm \omega \quad \text{where } \gamma = \frac{b}{2m} \quad \text{(as before)}$$

and $\omega = \sqrt{\gamma^2 - \omega_0^2}$ (note opposite sign!)

Therefore, the general solution is:

$$x(t) = A_+ e^{-(\gamma-\omega)t} + A_- e^{-(\gamma+\omega)t}$$

since $\gamma > \omega$, this is a sum of 2 decaying exponentials



2 possibilities: either it never crosses the equilibrium position (before settling there at last); or it crosses the equilibrium position exactly once before settling

The technical name for this case is "**overdamped**".

③ special case: **"critical damping"**

what if $\frac{b}{2m} = \omega_0$ exactly?

then $\alpha = -\frac{b}{2m} \pm 0 \rightarrow x(t) = Ae^{\alpha t}$

only one linearly independent solution!
can't solve for 2 initial conditions!
need a 2^{nd} solution!

Rule of thumb: if you have m "degenerate" (same) solutions of an n^{th} order linear diff eqn, then take the solns

$e^{\alpha t}, te^{\alpha t}, t^2 e^{\alpha t}, \ldots t^{m-1}e^{\alpha t}$

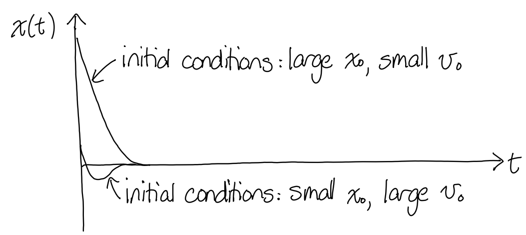
(note: of course we always have $m \le n$)

In this case, $m = n = 2$, so we just have $e^{\alpha t}$, $te^{\alpha t}$

can check that this is a solution by plugging in (HW)

$$\Rightarrow x(t) = Ae^{\alpha t} + Bte^{\alpha t}$$

This is the fastest damping solution, e.g. used for shocks on cars.



Similar to overdamping, the mass may cross the equilibrium position either zero or one times.

Critical damping: not qualitatively different from overdamping, just the limiting case which settles fastest.

Today's summary:

* review of complex #'s
* general algorithm for solving linear homogenous differential eqns
  - → n^{th} order eqn has n solutions of the form $e^{\alpha t}$ where α is complex
  - → in special case where $\alpha_i = \alpha_j$, use $e^{\alpha t}$ and $te^{\alpha t}$
* 2 different bases for solution space
  - $\{e^{i\omega t}, e^{-i\omega t}\}$ → useful for solving diff. eqn.
  - $\{\cos(\omega t), \sin(\omega t)\}$ → useful for initial conditions
* physical example: damped oscillator
  - → underdamped
  - → overdamped
  - → critically damped

  3 different physical behaviors all come out of the same algorithm for solving diff. eqns, just by checking all real & imaginary cases carefully

Next time:

* inhomogenous differential equations
* forced oscillator
* resonance

Reading for next time: H&L 1.7 or Georgi 2.2-2.4 or Morin 1.3