Reading: H&L 1.7 or Georgi 2.2-2.4 or Morin 1.3

Physics 15c (Hoffman)
Lecture #3
Thurs, Sept 9, 2010

Math: Inhomogenous Differential Equations

Physics: Driven Oscillator

Last time:

* review of complex #'s
* general algorithm for solving linear homogenous differential eqns
  → n^{th} order eqn has n solutions of the form $e^{\alpha t}$ where α is complex
  → in special case where $\alpha_i = \alpha_j$, use $e^{\alpha t}$ and $te^{\alpha t}$
* physical example: damped oscillator
  → underdamped
  $x(t) = e^{-\gamma t}[A\cos(\omega t) + B\sin(\omega t)]$ (oscillation at $\omega < \omega_0$ with a decaying exponential envelope)
  → overdamped
  $x(t) = Ae^{-(\gamma-\omega)t} + Be^{-(\gamma+\omega)t}$ (sum of 2 decaying exponentials)
  → critically damped
  $x(t) = Ae^{-\gamma t} + Bte^{-\gamma t}$ (fastest damping → limit of overdamped)
* 2 different solution bases: $\{e^{i\omega t}, e^{-i\omega t}\}$ and $\{\cos(\omega t), \sin(\omega t)\}$

  In different cases, it will be more convenient to use different solution basis vectors, e.g. use $e^{i\omega t}$ and $e^{-i\omega t}$ when 1^{st} solving diff eqn, but use $\cos(\omega t)$ and $\sin(\omega t)$ when finding initial conditions.

Goals for today:

* inhomogenous linear differential eqns
* forced oscillator example: driven RLC circuit

Inhomogenous linear differential equations:

We now know how to solve:

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_1\frac{dx}{dt} + a_0 x = 0 \quad \text{[eqn 1]}$$

But what about

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_1\frac{dx}{dt} + a_0 x = f(t) \quad \text{[eqn 2]}$$

inhomogenous! some arbitrary function $f(t)$ like a time-dependent external force on the system

Basic strategy:

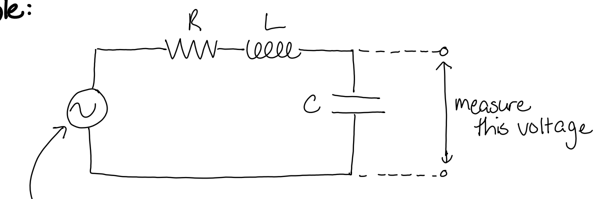
1. Solve for a particular solution $x_s(t)$ which satisfies eqn 2. This solution $x_s(t)$ is totally constrained (no free params)
   → it is the steady state solution after the influence of initial conditions has completely died away
2. Now remember linearity: means that if $x_1(t)$ is a solution to eqn 1 then $A_1 x_1(t)$ is a soln, and the whole soln to eqn 1 is:

   $x_t(t) = A_1 x_1(t) + \dots + A_n x_n(t)$

   This is a transient soln with n free params (A_i's) which are found using initial conditions.
3. If we plug $x_t(t)$ into eqn 2, we just get zero on the right side, so we can take our total soln to be:

   $$x(t) = \underbrace{x_s(t)}_{\text{steady state}} + \underbrace{x_t(t)}_{\text{transient (depends on initial conditions)}}$$

Example:



apply an AC voltage: $V(t) = V_0\sin(\omega_d t)$ [ω_d is the drive frequency]

Use Kirchoff's law to sum voltages around loop:

$$V_0 \sin(\omega_d t) = RI + L\frac{dI}{dt} + \underbrace{\frac{1}{C}q}$$

this is our measured voltage

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \sin(\omega_d t) \qquad [\text{eqn } 3]$$

Look for a particular solution → need a trick!
Notice that both sides of this eqn are real (b/c it does describe the real world!)
So if we add some purely imaginary terms to both sides, it won't change the real, physical part of the equation.
Basically, the strategy is to get everything in terms of complex exponentials, because it makes diff eqns easier to solve.
So what do we need to add to the R-side to get a complex exponential?
Notice that:

$$V_0 \sin(\omega_d t) = \mathrm{Im}(V_0 e^{i\omega_d t}) = \mathrm{Re}(-iV_0 e^{i\omega_d t})$$

So if we rewrite the R-side of eqn 3 as $-iV_0 e^{i\omega_d t}$ then all we have done is added an imaginary term $-iV_0\cos(\omega_d t)$ to the R-side.
Now on the L-side of eqn 3, we replace our real, physical charge $q(t)$ with a complex function $z(t) = q(t) + iq_i(t)$ where $\mathrm{Re}(z(t)) = q(t)$ is still the real, physical soln we care about but $q_i(t)$ is some throw-away imaginary part to make up for the $-iV_0\cos(\omega_d t)$ that we added to the R-side.

Ok, so we have:

$$L\frac{d^2z}{dt^2} + R\frac{dz}{dt} + \frac{1}{C}z = -iV_0 e^{i\omega_d t}$$

$$\Rightarrow \frac{d^2z}{dt^2} + \frac{R}{L}\frac{dz}{dt} + \frac{1}{LC}z = -i\frac{V_0}{L}e^{i\omega_d t}$$

as you will find on pset #1, $\omega_0^2 = \frac{1}{LC}$

this is a damping term
→ let's define $\Gamma \equiv \frac{R}{L}$ (similar to previous γ but differs by 2)

$$\Rightarrow \frac{d^2z}{dt^2} + \Gamma\frac{dz}{dt} + \omega_0^2 z = -i\frac{V_0}{L}e^{i\omega_d t}$$

Now we're in a better position to guess an exponential trial function:
Let $z(t) = z_0 e^{\alpha t}$ and plug in:

$$\alpha^2 z_0 e^{\alpha t} + \Gamma\alpha z_0 e^{\alpha t} + \omega_0^2 z_0 e^{\alpha t} = -i\frac{V_0}{L}e^{i\omega_d t}$$

$$\Rightarrow z_0 e^{\alpha t}(\alpha^2 + \Gamma\alpha + \omega_0^2) = -i\frac{V_0}{L}e^{i\omega_d t}$$

What is α? can't immediately cancel exponentials & solve...
→ another trick: "separation of variables"
(you will see this often in quantum mechanics)
→ put all t-dependence on same side of eqn

$$\underbrace{\frac{z_0(\alpha^2 + \Gamma\alpha + \omega_0^2)}{-iV_0/L}}_{\text{not time-dependent!}} = \frac{e^{i\omega_d t}}{e^{\alpha t}} = \underbrace{e^{(i\omega_d - \alpha)t}}_{\text{apparently time-dependent}}$$

how can this be? it can't!
→ R-side can't really be time-dependent
→ exponent must be zero

$$\Rightarrow \alpha = i\omega_d$$

Now we plug back in this value of $\alpha = i\omega_d$:

$$z_0 e^{i\omega_d t}(-\omega_d^2 + i\Gamma\omega_d + \omega_0^2) = -i\frac{V_0}{L}e^{i\omega_d t}$$

Finally the $e^{i\omega_d t}$ parts cancel:

$$z_0(-\omega_d^2 + i\Gamma\omega_d + \omega_0^2) = -i\frac{V_0}{L}$$

Remember, ω_d is our driving frequency → this is a fixed input to our system, as is V_0. Likewise, ω_0, Γ, and L are fixed parameters.

So the only unknown here that we should solve for is z_0:

$$z_0 = \frac{-iV_0/L}{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d}$$

$$\Rightarrow z(t) = \frac{-iV_0/L}{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d}e^{i\omega_d t}$$

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$$q(t) = Re(z(t)) = Re(z_0)\cos(\omega_d t) - Im(z_0)\sin(\omega_d t)$$

somewhat tedious & not terribly instructive...

$$z_0 = \frac{-iV_0/L}{\omega_0^2-\omega_d^2+i\Gamma\omega_d}\cdot\frac{\omega_0^2-\omega_d^2-i\Gamma\omega_d}{\omega_0^2-\omega_d^2-i\Gamma\omega_d}$$

$$= \frac{-\Gamma\omega_d V_0/L - iV_0/L(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}$$

$$\Rightarrow q(t) = \frac{-\Gamma\omega_d V_0/L}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\cos(\omega_d t) + \frac{V_0/L(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\sin(\omega_d t)$$

How does this really behave?
→ It's probably more instructive to look at $z(t)$
Let's assume damping is weak (R is small) and look at 3 different drive frequency regimes in the case of small damping.

① small frequency $\omega_d \ll \omega_0$

$$z(t) = \frac{-iV_0/L}{\omega_0^2-\omega_d^2+i\Gamma\omega_d}e^{i\omega_d t}$$

↖ this term in denominator dominates

$$z(t) \approx \frac{-iV_0/L}{\omega_0^2}e^{i\omega_d t}$$

$$q(t) = Re(z(t)) \approx \left(\frac{-iV_0/L}{\omega_0^2}\right)(i\sin(\omega_d t)) = \frac{V_0/L}{\omega_0^2}\sin(\omega_d t)$$

$$V(t) = \frac{q(t)}{C} \approx \frac{V_0}{LC\omega_0^2}\sin(\omega_d t) = V_0\sin(\omega_d t)$$

→ motion of charge & therefore measured voltage are nearly in phase with the driving voltage

② large frequency $\omega_d \gg \omega_0$

$$z(t) = \frac{-iV_0/L}{\omega_0^2-\omega_d^2+i\Gamma\omega_d}$$

↖ this term in denominator dominates

$$z(t) \approx \frac{-iV_0/L}{-\omega_d^2}e^{i\omega_d t}$$

$$q(t) = Re(z(t)) \approx \left(\frac{iV_0/L}{\omega_d^2}\right)(i\sin(\omega_d t)) = \frac{-V_0/L}{\omega_d^2}\sin(\omega_d t)$$

$$V(t) = \frac{q(t)}{C} \approx \frac{-V_0}{LC\omega_d^2}\sin(\omega_d t) = -\frac{\omega_0^2}{\omega_d^2}V_0\sin(\omega_d t)$$

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→ motion of charge & therefore measured voltage are nearly 180° out-of-phase w/ driving voltage

AND are <u>reduced</u> by a factor of ω_0^2/ω_d^2

(→ can build electronic frequency filters)

③ **resonance: $\omega_d = \omega_0$**

$$z(t) = \frac{-iV_0/L}{\underbrace{\omega_0^2-\omega_0^2}_{\text{cancel}}+i\Gamma\omega_0}e^{i\omega_0 t}$$

↖ this is the only non-vanishing term in denominator

$$z(t) = \frac{-V_0/L}{\Gamma\omega_0}e^{i\omega_0 t}$$

$$q(t) = Re(z(t)) = \frac{-V_0/L}{\Gamma\omega_0}\cos(\omega_0 t)$$

$$V(t) = \frac{q(t)}{C} = \frac{-V_0}{LC\Gamma\omega_0}\cos(\omega_0 t) = -V_0\frac{\omega_0}{\Gamma}\cos(\omega_0 t)$$

→ motion of charge & therefore measured voltage are exactly 90° out-of-phase w/ the driving voltage

AND are <u>amplified</u> by a factor of ω_0/Γ

For low resistance, Γ is small, and this results in a large gain → <u>RESONANCE</u>

Q=quality factor:

Define the "Q" = quality factor of an oscillator:

$$Q = \frac{\omega_0}{\Gamma} = \text{amplification factor at resonance}$$

Large Q → weakly damped, highly resonant

Small Q → strongly damped, muffled

Quiz: synthesize what we've learned from these 3 cases, and plot both the amplitude & phase response of the oscillator as a function of driving frequency ω_d.

Initial conditions

OK, so we have this particular steady-state solution:

$$q(t) = \mathrm{Re}\left[\frac{-iV_0/L}{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d} e^{i\omega_d t}\right]$$

But how can we satisfy initial conditions?
We need to add in the transient solution.

Recall: there are 3 cases for the general soln: $\begin{cases}\text{underdamped}\\ \text{overdamped}\\ \text{critically damped}\end{cases}$

The algebra is messy, so let's just take on one of these...

Underdamped ⇒ recall solution to homogenous eqn:

$$x_t(t) = e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \qquad \text{where } \omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\omega_0^2 - (\Gamma/2)^2}$$

$\gamma = \frac{b}{2m}$ for mass-on-spring
$= \frac{R}{2L}$ for circuit $= \frac{\Gamma}{2}$

⇒ Most general solution is:

$$q(t) = \mathrm{Re}\left[\frac{-iV_0/L}{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d} e^{i\omega_d t}\right] + e^{-\Gamma t/2}\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

Now we need to solve for A & B from initial conditions:

Example: turn on AC voltage at $t=0$, so $q(t=0)=0$, $I(t=0)=0$

$$q(t) = \frac{-\Gamma\omega_d V_0/L}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2}\cos(\omega_d t) + \frac{V_0/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2}\sin(\omega_d t)$$
$$+ e^{-\Gamma t/2}\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

$$q(t=0) = \frac{-\Gamma\omega_d V_0/L}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2} + A = 0$$

$$I(t) = \frac{\Gamma\omega\omega_d V_0/L}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2}\sin(\omega_d t) + \frac{\omega_d V_0/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2}\cos(\omega_d t)$$
$$-\frac{\Gamma}{2}e^{-\Gamma t/2}\left[A\cos(\omega t) + B\sin(\omega t)\right] + e^{-\Gamma t/2}\left[-\omega A\sin(\omega t) + \omega B\cos(\omega t)\right]$$

$$I(t=0) = \frac{\omega_d V_0/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2} - \frac{\Gamma}{2}A + \omega B = 0$$

$$\Rightarrow A = \frac{\Gamma\omega_d V_0/L}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2} \qquad B = \frac{\omega_d V_0}{L}\,\frac{\Gamma^2/2 - (\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2 + (\Gamma\omega_d)^2}$$

What happens to energy?

Mechanical energy is not conserved, because the resistor converts mechanical energy → heat

Energy consumed per unit time is $P = IV = I^2R = R\left\{\frac{dq}{dt}\right\}^2$

The power supply supplies the power (no surprise!)

→ work done by power supply is $P = IV = \frac{dq}{dt} V_0 \sin(\omega_d t)$

Energy consumed by resistor must equal work done by power supply...
→ but need to average over one full cycle to show this, b/c the L and C store the energy temporarily, so P_{in} from the power supply isn't instantaneously equal P_{out} to resistor

Let's see if we can confirm $E_R = E_{ps}$

total energy consumed by R in one full cycle ↗ ↖ total energy supplied by power supply in one full cycle

Unfortunately, we must do this in real #'s, b/c energy is <u>not</u> linear
→ when we square the full imaginary $z(t)$, we get spurious real terms.

$$E_R = \int_0^T R\left\{\frac{dq}{dt}\right\}^2 dt$$

$$= R\int_0^T \left\{ \frac{\Gamma\omega_d^2 V_0/L}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\sin(\omega_d t) + \frac{\omega_d V_0/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\cos(\omega_d t)\right\}^2 dt$$

$$= R\int_0^T \frac{\Gamma^2\omega_d^4 V_0^2/L^2}{[(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2]^2}\underbrace{\sin^2(\omega_d t)\,dt}_{\text{avg to } 1/2} + R\int_0^T \frac{\omega_d^2 V_0^2/L^2\,(\omega_0^2-\omega_d^2)^2}{[(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2]^2}\underbrace{\cos^2(\omega_d t)\,dt}_{\text{avg to } 1/2}$$

$$+ R\int_0^T \frac{2\Gamma\omega_d^3 V_0^2/L^2\,(\omega_0^2-\omega_d^2)}{[(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2]^2}\underbrace{\cos(\omega_d t)\sin(\omega_d t)\,dt}_{\text{avg to } 0}$$

$$= R\frac{T}{2}\left\{\frac{\Gamma^2\omega_d^4V_0^2/L^2+\omega_d^2V_0^2/L^2(\omega_0^2-\omega_d^2)^2}{[(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2]^2}\right\} = \frac{RT}{2}\frac{V_0^2}{L^2}\frac{\omega_d^2}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}$$

$$E_{ps} = \int_0^T V_0\sin(\omega_d t)\left\{\frac{dq}{dt}\right\}dt$$

$$= \int_0^T V_0\sin(\omega_d t)\left\{\frac{\Gamma\omega_d^2V_0/L}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\sin(\omega_d t) + \frac{\omega_d V_0/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\cos(\omega_d t)\right\}dt$$

$$= \int_0^T \frac{\Gamma\omega_d^2V_0^2/L}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\underbrace{\sin^2(\omega_d t)\,dt}_{\text{avg to } 1/2} + \int_0^T \frac{\omega_d V_0^2/L\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}\underbrace{\sin(\omega_d t)\cos(\omega_d t)\,dt}_{\text{avg to } 0}$$

$$= \frac{T}{2}\frac{\Gamma\omega_d^2V_0^2/L}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2} \qquad \text{remember } \Gamma \equiv \frac{R}{L}$$

$$= R\frac{T}{2}\frac{V_0^2}{L^2}\frac{\omega_d^2}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}$$

Good! $E_{ps} = E_R$ ⇒ work-energy theorem is indeed true!

We can also plot the limiting behavior of energy input as a function of driving frequency, as we did for amplitude & phase.

$$\langle P\rangle = \frac{\text{energy in one cycle}}{T} = \frac{R}{2}\frac{V_0^2}{L^2}\frac{\omega_d^2}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}$$

remember $\omega_0 = \sqrt{\frac{1}{LC}}$ ⇒ $\langle P\rangle = \frac{1}{2}V_0^2RC^2\frac{\omega_0^4\omega_d^2}{(\omega_0^2-\omega_d^2)^2+(\Gamma\omega_d)^2}$

$\omega_d \ll \omega_0$: $\langle P\rangle \propto \frac{\omega_0^4\omega_d^2}{\omega_0^4[(1-\omega_d^2/\omega_0^2)^2+\Gamma^2\omega_d^2/\omega_0^4]} \approx \omega_d^2$

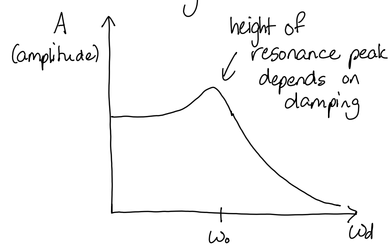
$\omega_d \gg \omega_0$: $\langle P\rangle \propto \frac{\omega_0^4\omega_d^2}{\omega_d^4[(1-\omega_0^2/\omega_d^2)^2+\Gamma^2/\omega_d^2]} \approx \frac{\omega_0^4}{\omega_d^2}$

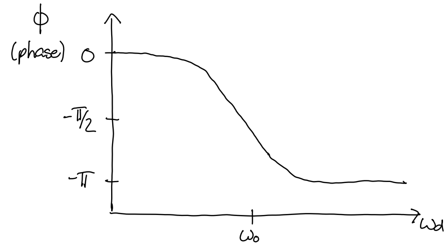
$\omega_d = \omega_0$: $\langle P\rangle \propto \frac{\omega_0^4\omega_d^2}{\Gamma^2\omega_d^2} = \frac{\omega_0^4}{\Gamma^2} = \omega_0^2Q^2$

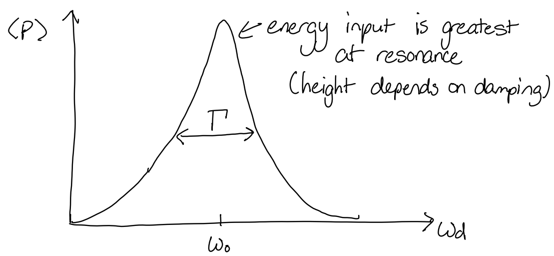
where $Q \equiv \frac{\omega_0}{\Gamma}$ = "quality factor" of the oscillator
= how easily it rings up at resonance, inversely proportional to damping

Frequency dependence of A, ϕ, and $\langle P \rangle$

Put it all together:







Summary:

* solved inhomogenous linear differential equation
  - → full solution is steady state + transient
* solved for transient coefficients from initial conditions
* example: forced, damped oscillator
  - → resonance
  - → energy conservation

Next time:

* coupled oscillators

Reading for next time: not covered at all in H & L

try reading Georgi 3.1-3.3 and 4.1
or
Morin 2.1-2.2