Physics 15c (Hoffman)
Lecture # 9
Tues, Oct 5, 2010

Reading for today:
H&L, chapter 5
or Morin section 5.2

Sound Waves

Goals for today:

Write down an expression for a sound wave, understand its properties:

* decibels, dynamic range
* bulk modulus
* impedance
* sound velocity

Sound = longitudinal pressure wave

closely bunched molecules → high P

rarefied molecules → low P

So we could consider the displacements of the individual air molecules, or the magnitude of the pressure changes.

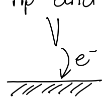
But it turns out the human ear perceives "loudness" as the total intensity of the sound wave hitting it.

intensity = $\frac{\text{power}}{\text{area}}$ = energy passing through a unit area per unit time

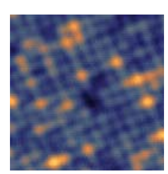
Human hearing: **AMAZING** "dynamic range"

"dynamic range" = ratio of largest detectable input to smallest detectable input

In my lab, I use a scanning tunneling microscope to detect very small tunneling currents between an atomically sharp tip and a sample of interest:



5nm × 5nm image of high-T_c superconductor $Bi_2Sr_2CaCu_2O_8$



I need to detect currents < 1 picoAmp
⇒ I buy the best current amplifiers that \$ can buy.
The dynamic range on these guys is 10^9.

The dynamic range of the human ear is 10^{12} !!!

Softest sound we can hear: $I_{min} = 10^{-12}\ W/m^2$

Loudest sound (pain threshold): $I_{max} = 1\ W/m^2$

decibels = dB = measure of "loudness" or "sound volume"

$$\beta(dB) = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

I_0 = threshold = $10^{-12}\ W/m^2$

normal conversation: 50-60 dB
(normal IPOD users: 60-70 dB
→ generation of early deafness predicted)

Energy carried by a wave (review)

$$\boxed{\frac{dE_K}{dx} = \frac{1}{2}\rho_e\left(\frac{\partial\xi(x,t)}{\partial t}\right)^2} \qquad \boxed{\frac{dE_S}{dx} = \frac{1}{2}E\left(\frac{\partial\xi(x,t)}{\partial x}\right)^2}$$

Look in more detail at the total energy for a specific non-dispersive wave $\xi_0\cos(kx-\omega t)$ [where $\omega = c_w k$]

$$\Rightarrow \frac{dE_{tot}}{dx}(x,t) = \frac{dE_K}{dx}(x,t) + \frac{dE_S}{dx}(x,t)$$
$$= \rho_e\left(\frac{\partial}{\partial t}\xi_0\cos(kx-\omega t)\right)^2 = \rho_e\omega^2\xi_0^2\sin^2(kx-\omega t)$$

The velocity at which the wave (& therefore energy) moves is c_w
⇒ the energy flowing through a given point is

$$\frac{dE}{dt} = c_w\frac{dE}{dx}$$

Let's look at some averages to understand the big picture of where the energy is coming from & where it is going.

③

Energy density $\frac{dE}{dx} = \rho_\ell \omega^2 \xi_0^2 \sin^2(kx-\omega t)$

$\rightarrow$ averages to $\left\langle \frac{dE}{dx} \right\rangle = \frac{1}{2}\rho_\ell \omega^2 \xi_0^2$ } average energy density of a wave

This all travels at $c_w = \sqrt{E/\rho_\ell}$ ($\leftarrow E$ = elastic constant here)

$\rightarrow$ the rate of energy transfer (in Joules/second = Watts) is:

$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} c_w \rho_\ell \omega^2 \xi_0^2 = \frac{1}{2}\underbrace{\sqrt{E\rho_\ell}}_{\text{what is this?}}\,\omega^2 \xi_0^2$ } average power of a wave

$\sqrt{E\rho_\ell}$ is called the "impedance" of the medium (often called Z)

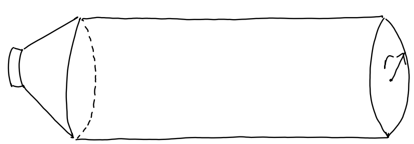
Note: impedance is a property of the medium, not of the wave itself.

How does this energy calculation apply to sound?

Generic longitudinal wave: $P = \frac{dE}{dt} = \sqrt{E\rho_\ell}\,\omega^2 \xi_0^2 \sin^2(kx-\omega t)$

what do these terms mean for sound?

Imagine a round speaker attached to a pipe of radius r:



Air (room temperature & pressure) has mass density 1.29 kg/m³

$\rightarrow$ linear mass density $\rho_\ell = 1.29\,\pi r^2$ kg/m

What about elastic modulus? What __is__ an elastic modulus, really?

k_s = spring constant (N/m) = force required for unit change in length

k_s scales inversely with length of spring

(easier to stretch a long spring than a short one)

$\rightarrow$ $k/(1/L)$ should be constant $\equiv$ __E__

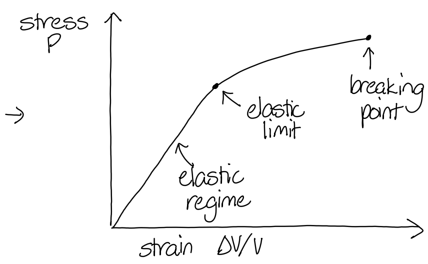
④

But for a 3-dimensional material, it makes more sense to talk about a pressure change ΔP required to bring about a unit change in volume ΔV.

But again, for a larger volume to begin with, it should be easier to change the total volume, i.e. $\Delta P/\Delta V$ should scale inversely with total volume V.

$\Rightarrow$ bulk modulus $= B = \frac{-\Delta P}{\Delta V/V} = \frac{\text{pressure change required}}{\text{fractional change in volume}}$

typical solid $\rightarrow$



(Note: weird convention y-axis is independent variable... this is the way engineers do it.)

Gases: a little more complicated than solids

$\rightarrow$ need to use ideal gas law ... plus some careful thought

$PV = NRT$ ← temperature (K)

pressure (Pa); volume (m³); # of moles; gas constant = 8.314 J/(mol·K)

suggests that $P \propto \frac{1}{V}$

but this is over-simplified & wrong

Temperature goes up when a gas is compressed.

(need to do work to compress it $\rightarrow$ energy must go somewhere!)

$\rightarrow$ rising T increases P even more than just the geometric compression alone

$P \propto \frac{1}{V^\gamma}$ where $\gamma = \frac{\text{specific heat at constant } P}{\text{specific heat at constant } V} = \frac{C_P}{C_V} > 1$

(you will derive this in physics 181 or chem 161)

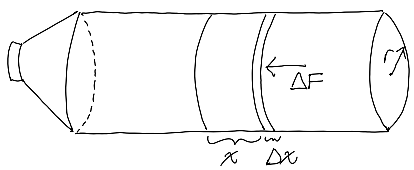
$\gamma = 5/3$ for monatomic gases (He, Ne, etc.)
$\gamma = 7/5$ for diatomic gases (O_2, N_2, etc.)

$\Rightarrow$ use $\gamma = 7/5$ because air is ~20% O_2, ~80% N_2

We defined the bulk modulus: $B = \frac{-\Delta P}{\Delta V/V}$ (minus sign because $P\uparrow, V\downarrow$)

$$B = -V\frac{dP}{dV} = -V\frac{d}{dV}\left(\frac{\text{const.}}{V^\gamma}\right) = -V\left(-\gamma\frac{\text{const.}}{V^{\gamma+1}}\right) = \frac{\gamma\cdot\text{const.}}{V^\gamma} = \gamma P$$

Back to our 1-dim example:



$\rho_e = \rho\pi r^2$

our old one-dimensional mass-spring elastic constant

$$B = \frac{-\Delta P}{\Delta V/V} = \frac{-F/(\pi r^2)}{(\pi r^2\Delta x)/(\pi r^2 x)} = \frac{-F}{\Delta x/x}\cdot\frac{1}{\pi r^2} = \frac{E}{\pi r^2} \Rightarrow E = B\cdot\pi r^2$$

$$v_{sound} = \sqrt{\frac{E}{\rho_e}} = \sqrt{\frac{B\pi r^2}{\rho\pi r^2}} = \sqrt{\frac{B}{\rho}} = \text{fixed property of air}$$

$$Z = \sqrt{E\rho_e} = \sqrt{B\pi r^2\,\rho\pi r^2} = \underbrace{\sqrt{B\rho}\,(\pi r^2)}$$

It seems that the impedance is geometry-dependent - what does this mean? Conceptually, impedance = force/velocity. But the force required to bring about a certain velocity must be dependent on the cross-sectional area which is to be set in motion at that velocity. So it often makes more sense to talk about a geometry-independent impedance:
"characteristic impedance" of a material
$= Z/(\text{surface area through which wave propagates}) = \sqrt{B\rho} \equiv Z_0$

Energy transfer rate is: $\left\langle\frac{dE}{dt}\right\rangle = \frac{1}{2}\sqrt{B\rho}\,(\pi r^2)\,\omega^2\xi_0^2$

Intensity (what we hear) is: $\frac{\left\langle\frac{dE}{dt}\right\rangle}{\text{area}} = \frac{1}{2}\underbrace{\sqrt{B\rho}}\,\omega^2\xi_0^2$ ← displacement amplitude

fixed property of the air $= \sqrt{\gamma P\rho}$

frequency of sound

Example: calculate the average power & intensity

a) 2 inch speaker ($r = 0.025$ m) making ± 1mm oscillations at 1 kHz ($\omega = 6.28\times10^3$ rad/s)

b) 12 inch speaker ($r = 0.15$ m) making ± 5 mm oscillations at 20 Hz ($\omega = 126$ rad/s)

Answer: $\langle\text{power}\rangle = \left\langle\frac{dE}{dt}\right\rangle = \frac{1}{2}\sqrt{B\rho}\,(\pi r^2)\,\omega^2\xi_0^2$

$$\langle\text{intensity}\rangle = \frac{\left\langle\frac{dE}{dt}\right\rangle}{\text{area}} = \frac{1}{2}\sqrt{B\rho}\,\omega^2\xi_0^2$$

So we need to plug in some numbers:

$B = \gamma P = \frac{7}{5}(101\text{ kPa}) = 1.4\times10^5$ Pa

$\rho = 1.29$ kg/m^3

$Z_0 = \sqrt{B\rho} = \sqrt{(1.4\times10^5\text{ N/m}^2)(1.29\text{ kg/m}^3)} = 427$ kg/(m^{2}s)

$v_{sound} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{(1.4\times10^5\text{ N/m}^2)}{(1.29\text{ kg/m}^3)}} = 331$ m/s

a) $\langle\text{power}\rangle = \frac{1}{2}(427\text{ kg/m}^2\text{s})\,\pi(0.025\text{m})^2(6.28\times10^3\text{ rad/s})^2(0.001\text{m})^2 = 16.5$ W

$\langle\text{intensity}\rangle = \frac{16.5\text{ W}}{\pi(0.025\text{m})^2} = 8403$ W/m^2

$= 10\log_{10}\left(\frac{8403\text{ W/m}^2}{10^{-12}\text{ W/m}^2}\right) = 159$ dB

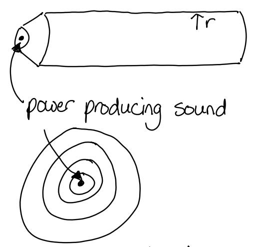
b) $\langle\text{power}\rangle = \frac{1}{2}(427\text{ kg/m}^2\text{s})\,\pi(0.15\text{ m})^2(126\text{ rad/s})^2(0.005\text{m})^2 = 5.99$ W

$\langle\text{intensity}\rangle = \frac{5.99\text{W}}{\pi(0.15\text{m})^2} = 84.7$ W/m^2

$= 10\log_{10}\left(\frac{84.7\text{ W/m}^2}{10^{-12}\text{ W/m}^2}\right) = 139$ dB

Sound in 3 dim:

1-dim



$I = \frac{P}{\pi r^2}$ ← radius of pipe

doesn't depend on distance from source

3-dim

$I = \frac{P}{4\pi r^2}$ ← distance from source

$\Rightarrow$ if sound spreads isotropically, intensity falls of as $1/(\text{distance})^2$

Example: Firebell & decibel meter

Suppose the sound intensity of a firebell is 98 dB at 1m away. How loud is it at 2m?

Answer: Sound in 3dim falls off like $1/(\text{distance})^2$
→ if we're twice as far away then it's 1/4 as loud

$\beta(dB) = 10 \log_{10}\left(\frac{I}{I_0}\right)$ where $I_0 = 10^{-12}\ W/m^2$

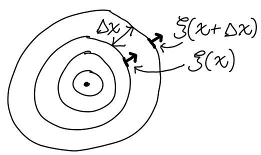
$\beta_1 = 98\ dB = 10\log_{10}(I_1/I_0) \Rightarrow I_1 = I_0\, 10^{9.8}$

$\beta_2 = 10\log_{10}(I_2/I_0) = 10\log_{10}(10^{9.8}/4) = 98 - 10\log_{10}(4) = 92\ dB$

Displacement & pressure variation:

How can we relate I to displacement amplitude ξ_0 and pressure variation Δp?

$I = \frac{1}{2}\sqrt{B\rho}\,\omega^2 \xi_0^2 \Rightarrow \boxed{\xi_0 = \frac{1}{\omega}\sqrt{\frac{2I}{Z_0}}}$



$$\frac{\Delta V}{V} = \frac{[\xi(x+\Delta x) - \xi(x)](\text{surface area of sphere})}{\Delta x(\text{surface area of sphere})} = \frac{d\xi}{dx}$$

$$\Delta p = -B\frac{\Delta V}{V} = -B\frac{d\xi}{dx}$$

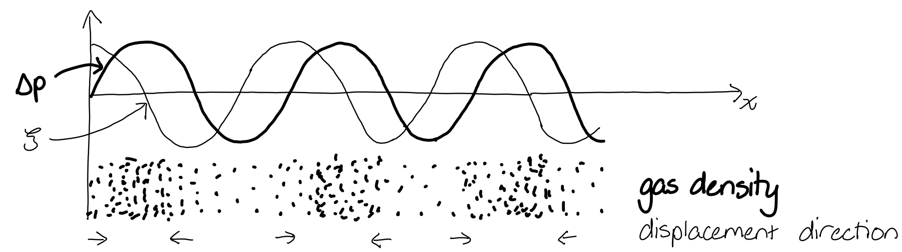
Recall: $Z_0 = \sqrt{B\rho}$ and $v_{sound} = \sqrt{B/\rho}$
$\Rightarrow B$ can be written as $Z_0 v_{sound}$
$\Rightarrow \Delta p = Z_0 v_{sound}\frac{d\xi}{dx}$

If we start with a specific waveform $\xi(x,t) = \xi_0 \cos(kx - \omega t)$ then we can investigate the phase relationship between pressure and displacement.

$\frac{d\xi}{dx} = -k\xi_0 \sin(kx - \omega t) = -\frac{\omega}{v_{sound}}\xi_0 \sin(kx-\omega t)$

$\Rightarrow \Delta p = -Z_0 v_{sound}\frac{d\xi}{dx} = Z_0 \omega \xi_0 \sin(kx - \omega t)$

displacement & pressure are $\pi/2$ out of phase



Sound velocity Compare: $v_{sound} = \sqrt{\frac{B}{\rho}}$ and $v_{\text{long wave on spring}} = \sqrt{\frac{E}{\rho_e}}$

velocity ↑ as restoring force ↑
↓ as mass (inertia) ↑

velocity of a wave $\propto \sqrt{\frac{\text{restoring force property}}{\text{inertia property}}}$

Summary:

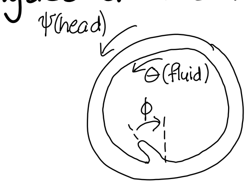
* sound = longitudinal pressure wave
* dynamic range = ratio of largest detectable input to smallest detectable input
* human ear perceives wave intensity = power / area
* decibels = $10 \log\left(\frac{I}{I_0}\right)$ where $I_0 = 10^{-12}\ W/m^2$ = minimum detectable intensity
* average power carried by a wave: $\langle P \rangle = \frac{1}{2} Z \omega^2 \xi_0^2$
* impedance of a medium: $Z = \sqrt{E \rho_e}$
* equation of state: $P \propto \frac{1}{V^\gamma}$ where $\gamma = \frac{\text{specific heat at constant } P}{\text{specific heat at constant } V} = \frac{C_P}{C_V} > 1$
* bulk modulus $= B = \frac{\text{stress}}{\text{strain}} = \frac{-\Delta P}{\Delta V/V} = \gamma P$
* $v_{sound} = \sqrt{\frac{B}{\rho}}$; $Z_0 = \sqrt{B\rho}$
* intensity $I = \frac{1}{2} Z_0 \omega^2 \xi_0^2 \Rightarrow \xi_0 = \frac{1}{\omega}\sqrt{\frac{2I}{Z_0}}$

  $\Delta p = \sqrt{2 I Z_0}$

Next time: Doppler effect & shock waves

Reading for next time: Hirose & Lonngren, chapter 8 or Morin section 7.2 (or a more advanced & detailed treatment of shock waves in Georgi chapter 14)

Physics of Dizziness



cupula is deflected by motion of fluid in semi-circular canal, in response to angular motion of head

Like the geophone, on HW #2:

$\Sigma F = m\, a_{tot}$ ← total acceleration of cupula = acceleration of cupula w.r.t. head + acceleration of head w.r.t. world

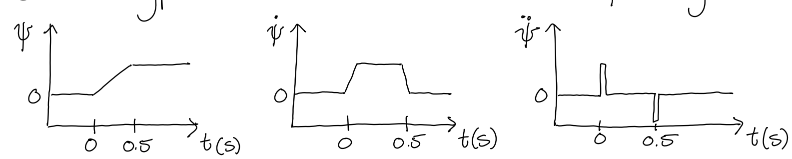
this is a generalized inertia term (effective mass of cupula)

$\Sigma F = -b\dot\phi - k\phi = m(\ddot\phi + \ddot\psi)$ [Note: in real ear, damping ≫ restoring force]

$\Rightarrow \ddot\phi + \frac{b}{m}\dot\phi + \frac{k}{m}\phi = -\ddot\psi$ ← acceleration of the head

What does acceleration of head look like?

→ consider typical turn of head in one direction, lasting ~1/2 sec



How do we solve this equation?

Step 1: solve "transient", "homogenous" eqn with $\ddot\psi = 0$

$$\ddot\phi + \frac{b}{m}\dot\phi + \frac{k}{m}\phi = 0$$

ansatz: $\phi(t) = e^{\alpha t}$

$$\alpha^2 + \frac{b}{m}\alpha + \frac{k}{m} = 0$$

$$\alpha = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{4k}{m}}}{2} = -\frac{b}{2m} \pm \frac{b}{2m}\sqrt{1 - \frac{4km}{b^2}}$$

$$\alpha = -\frac{b}{2m}\left(1 \mp \underbrace{\sqrt{1-\frac{4km}{b^2}}}\right)$$

$F_{damp} \gg F_{restore}$

$\Rightarrow b \gg \sqrt{\frac{k}{m}} \Rightarrow$ this term is small!

$$\alpha = -\frac{b}{2m}\left(1 \mp \left(1-\frac{2km}{b^2}\right)\right)$$

$$= \underbrace{-\frac{b}{m}}_{\text{large}} \quad \text{or} \quad -\frac{b}{2m}\left(\frac{2km}{b^2}\right) = \underbrace{-\frac{k}{b}}_{\text{small}}$$

$\Rightarrow \phi(t) = Ae^{-\frac{b}{m}t} + Be^{-\frac{k}{b}t}$

= sum of 2 decaying exponentials

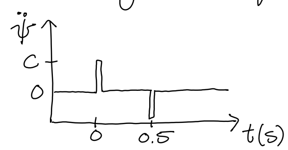
with $T_1 = \frac{m}{b} \approx 5$ ms, $T_2 = \frac{b}{k} \approx 10$ s } measured empirically for typical human ear

T_1 = time constant to reach terminal velocity (dictated by b)

T_2 = time constant to return to equilibrium position (dictated by k)

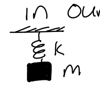
Step 2: solve "particular solution", "inhomogenous" equation with $\ddot{\psi} \neq 0$

take $\ddot{\psi} = c$ (constant)



$$\ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{k}{m}\phi = -c$$

↖ Note: this just acts like gravity in our typical spring problem:



$\Rightarrow$ c acts only to re-center the new equilibrium of the cupula

Note that we have 3 effective forces on the cupula in this case, and they turn out to be very different in magnitude:

$$mc \gg b\dot{\phi} \gg k\phi$$

effective force from rapid acceleration of head ↗ ; damping force ; restoring force

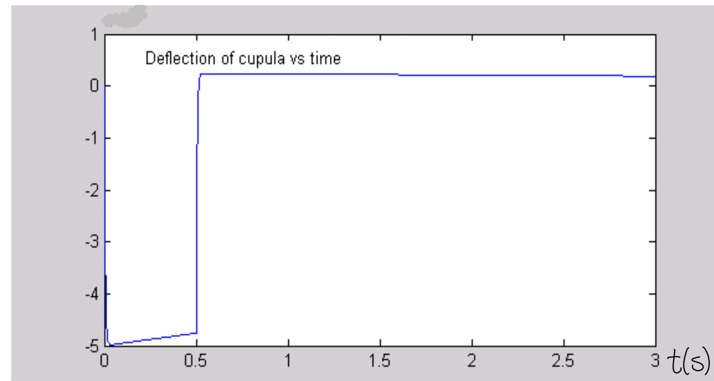
terminal velocity → v_T for cupula to move to new position is large, since c at typical head turn frequencies 0.1 – 10 Hz is $\gg$ acceleration from the restoring force or damping force

$\Rightarrow$ cupula rapidly reaches new equilibrium position

As $\ddot{\psi} = 0$ again, cupula slowly decays to original equilibrium with $T_2 \approx 10$ s (damping force dominates over restoring force)

Then when $\ddot{\psi} = -c$ (which usually occurs only ~0.5s later), there's a rapid return the rest of the way back to the original equilibrium, pulled mostly by the motion of the head, not by k.

$\frac{1}{2}$ second head turn



Question: what happens when you spin for 60 seconds?

spin for 60s