Physics 15c (Hoffman)
Lecture #13
Tues, Oct 19, 2010

Multi-dimensional boundary conditions

Last time:

* Reflection:
  - qualitative: fixed vs. free ends
    - fixed → reflected up-side-down
    - free → reflected right-side-up
  - quantitative: R and T determined by impedance
    - R → reflection coefficient
    - T → transmission coefficient

$$\xi_R = \frac{Z_1 - Z_2}{Z_1 + Z_2}\xi_I \qquad \xi_T = \frac{2Z_1}{Z_1+Z_2}\xi_I$$

$$Z = \text{impedance} = \frac{\text{force}}{\text{velocity}}$$

* multi-dim waves: $\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

  where $\vec{k}$ = wavevector = direction of propagation
  dispersion relation: $\omega = c_w|\vec{k}|$
  (infinite # of $\vec{k}$'s for each ω)

Goals for today:

* derivation of wave eqn on a drum head
* boundary conditions in multi-dimensions
* Can you hear the shape of a drum?

Fourier Transforms in $d \geq 2$

We have plane wave solutions for travel in arbitrary direction:

$$\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \quad \text{where} \quad \omega = c_w|\vec{k}|$$

Note: there are now **infinitely** many allowed $\vec{k}$ values for every ω. The constraint imposed by the dispersion relation $\omega = c_w|\vec{k}|$ is on the magnitude of $\vec{k}$, but any direction is allowed.

Each of these $\xi(\vec{x},t) = \xi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ with appropriate ω and $\vec{k}$ is a normal mode solution to the continuous wave equation:

$$\frac{\partial^2 \xi(\vec{x},t)}{\partial t^2} = c_w^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\xi(\vec{x},t) = c_w^2 \nabla^2 \xi(\vec{x},t)$$

As with all normal modes we have looked at, <u>any</u> arbitrary wave of arbitrary shape can be expressed by a linear combination of these normal modes
→ they are **complete**
→ they **span** the vector space of all possible solutions to the wave equation

Extension of Fourier transform to multi-dim:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)e^{i(k_x x + k_y y)}\,dk_x\,dk_y$$

$$= \int \tilde{F}(\vec{k})e^{i\vec{k}\cdot\vec{x}}\,d^2k \qquad \text{(same formula as above, just shorthand)}$$

$$\tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-i(k_x x + k_y y)}\,dx\,dy$$

$$= \frac{1}{(2\pi)^2}\int f(x,y)e^{-i\vec{k}\cdot\vec{x}}\,d^2x$$

Wave propagation from initial conditions:

If a wave has form $f(x,y)$ at $t=0$

→ then the Fourier integral can break it into components of the form $e^{i\vec{k}\cdot\vec{x}}$

→ and each component will travel as: $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$
(where ω depends on $\vec{k}$ via dispersion relation)

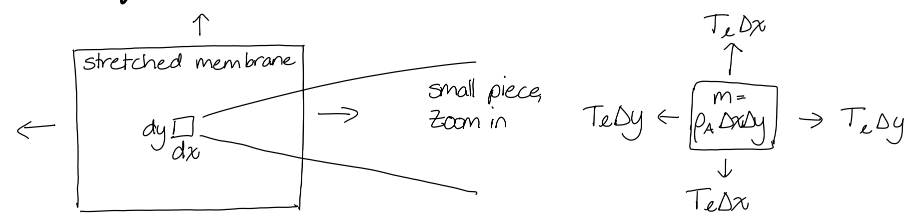
→ so if we know the initial shape of a wave pulse $f(x,y)$ then we know how it travels for all time

Quiz: Suppose $f(x,y) = e^{-(x^2+y^2)/2}$ at $t=0$

on a medium that satisfies $\frac{\partial^2 \xi}{\partial t^2} = c_w^2 \nabla^2 \xi$

Write an expression for the full $\xi(x,y,t)$ for all $t>0$
(don't bother to explicitly compute any integrals).

Wave eqn on a stretched membrane:



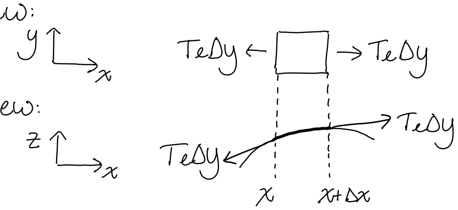
Increment is pulled from 4 sides by linear tension T_e
(T_e is tension per length, and force is proportional to the length of edge over which the tension can act.)

Forces are balanced in the xy plane
(Like a string! We found that T must be constant along the axis of the string in the small amplitude = small angle approximation.)

Consider vibrations only along the z-axis.

Top view:

Side view:



We get a force on the z-axis just like the string:

$$F_z = -T_e \Delta y \frac{\partial z(x,y,t)}{\partial x} + T_e \Delta y \frac{\partial z(x+\Delta x, y, t)}{\partial x}$$

$$\approx T_e \frac{\partial^2 z(x,y,t)}{\partial^2 x} \Delta x \Delta y$$

Do the same thing with the other edge (i.e. look at $z \uparrow \!\!\rightarrow y$):

$$F_{z_{\text{total}}} = T_e \frac{\partial^2 z(x,y,t)}{\partial x^2} \Delta x \Delta y + T_e \frac{\partial^2 z(x,y,t)}{\partial^2 y} \Delta x \Delta y$$

Equation of motion:

$$\rho_A \Delta x \Delta y \frac{\partial^2 z(x,y,t)}{\partial t^2} = T_\ell \frac{\partial^2 z(x,y,t)}{\partial x^2}\Delta x \Delta y + T_\ell \frac{\partial^2 z(x,y,t)}{\partial^2 y}\Delta x \Delta y$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = \frac{T_\ell}{\rho_A}\left\{ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right\}$$

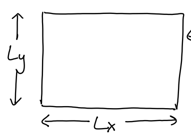
Use normal modes: $z(x,y,t) = z_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

plug in $\rightarrow$ dispersion $\omega = \sqrt{\frac{T_\ell}{\rho_A}}\,|\vec{k}|$

$\Rightarrow v_{phase} = \frac{\omega}{|\vec{k}|} = \sqrt{\frac{T_\ell}{\rho_A}}$ where $\begin{cases} T_\ell = \text{tension per unit length} \\ \rho_A = \text{areal mass density} \end{cases}$

Boundary conditions for d=2

Let's clamp the edges of our membrane, to make a drum head. For simplicity, let's make it a rectangular drum head.



membrane is clamped to $z=0$ on the edges
$\Rightarrow$ nodes in spatial wave function

$$z(0,y,t) = z(L_x,y,t) = z(x,0,t) = z(x,L_y,t) = 0$$

Standing wave (educated guess):

$$z(x,y,t) = \sin(k_x x)\sin(k_y y)\underbrace{e^{-i\omega t}}$$

or cos or sin
(depends on the phase, i.e. just depends on when $t=0$)

[Note: standing wave in 2-dim can be shown to be a sum of 4 traveling waves:

$$e^{i(k_x x + k_y y - \omega t)} - e^{i(-k_x x + k_y y - \omega t)} - e^{i(k_x x - k_y y - \omega t)} + e^{i(-k_x x - k_y y - \omega t)}$$

$$= -4\sin(k_x x)\sin(k_y y)\,e^{-i\omega t}$$
]

Need to ensure that $z(x,y,t) = \sin(k_x x)\sin(k_y y)e^{i\omega t}$ has spatial nodes at $x = 0, L_x$; $y = 0, L_y$

Because we juditiously chose to work with sines instead of cosines, we've already automatically satisfied $z(0,y,t)=0$ and $z(x,0,t)=0$.

So we just need to require that:

$$z(L_x,y,t) = 0 \Rightarrow \sin(k_x L_x) = 0$$

$$\Rightarrow k_x = \frac{\pi n}{L_x} \quad \text{(integer } n\text{)}$$

$$z(x,L_y,t) = 0 \Rightarrow \sin(k_y L_y) = 0$$

$$\Rightarrow k_y = \frac{\pi m}{L_y} \quad \text{(integer } m\text{)}$$

$$\Rightarrow \omega = \sqrt{\frac{T_\ell}{\rho_A}}\sqrt{\left(\frac{\pi n}{L_x}\right)^2 + \left(\frac{\pi m}{L_y}\right)^2} = \pi\sqrt{\frac{T_\ell}{\rho_A}}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}$$

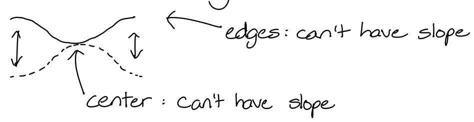
Harmonics are not nice integer multiples
$\rightarrow$ drum does not have a clear pitch

Most musical instruments are linear 1-dim waves!

Quiz: Assume $L_x = L_y = L$
What are the lowest 5 frequencies?
Even in this high-symmetry case, do we get exact harmonics?

Chladni plate: "drum" with fixed center & free boundary
(normal drum has fixed boundary & free center)

side view:



Summary:

* multi-dim Fourier transforms:

2-dim
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x,k_y)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$
$$\tilde{F}(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

general d
$$f(\vec{x}) = \int \tilde{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d^d k$$
$$F(\vec{k}) = \frac{1}{(2\pi)^d}\int f(x,y)\, e^{-i\vec{k}\cdot\vec{x}}\, d^d x$$

* normal modes on a drum: $\omega = \pi\sqrt{\frac{T_e}{\rho_A}}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}$

(not even harmonics!)

Next time:

LC transmission lines → lead-in to E&M waves

Quiz Answers

Wave propagation from initial conditions

$$\tilde{F}(\vec{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{-(x^2+y^2)/2}\, e^{-i(k_x x + k_y y)}$$

(Note: you could do this integral by completing the square in x and y, shifting the origin, then proceeding in polar coords.)

$$\xi(x,y,t) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y\; \tilde{F}(\vec{k})\, e^{i(k_x x + k_y y - c_w|\vec{k}|t)}$$

(Note on the relative sign of k_x, k_y and ω: the sign choice for k_x, k_y is arbitrary as long as it is opposite for the forward & inverse FT's. Sign for ω is chosen so wave spreads out as t increases.)

Square Drum

$$\omega = \pi\sqrt{\frac{T_e}{\rho_A}}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2} \;\Rightarrow\; f = \frac{1}{2L}\sqrt{\frac{T_e}{\rho_A}}\,(n^2+m^2)^{1/2}$$

n	m	f (in units of $\frac{1}{2L}\sqrt{T_e/\rho_A}$)
1	1	$\sqrt{2}$
1	2	$\sqrt{5}$
2	2	$2\sqrt{2}$
1	3	$\sqrt{10}$
2	3	$\sqrt{13}$

(Note that $(n,m) = (2,1)$ or $(3,1)$ produce degenerate frequencies & are thus not listed.)

Even in this high-symmetry case we do not have even harmonics.