Physics 15c (Hoffman)
Lecture # 11
Tues, Oct 12, 2010

Musical Instruments: standing waves, waves on strings

Where are we in this course?
Covered most of the mathematics of waves:
- complex numbers
- differential eqns
- Fourier transforms

Covered some general properties of all waves
- dispersion relation
- wave velocities (v_{phase} vs. v_{group})
- information, energy transmission
- Doppler shift (different for waves in matter/vacuum)

Now we're into the examples part of the course, where we discuss various physical phenomena associated with actual waves. We'll cover:
- sound & boundary conditions
- LC transmission lines
- electromagnetic waves
- reflection, refraction, interference, diffraction
- geometrical optics
- modern examples

Last time:

Doppler effect
* matter waves: must take into account 3 velocities
  → velocity of source, observer, medium

$$f' = \frac{(c_w + v_m) - v_o}{(c_w + v_m) - v_s} f_o$$

(v_o = observer, v_s = source, and v_m = medium, e.g. wind)

* light waves: only one relevant velocity
  → velocity of observer w.r.t. source

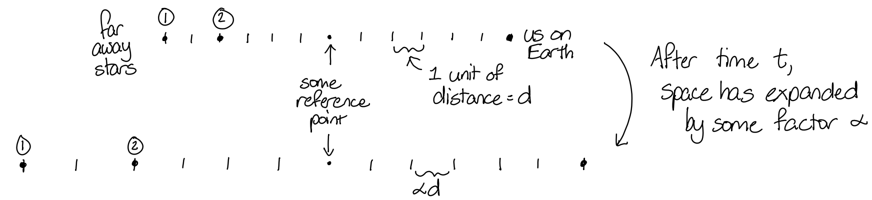
$$f' = \sqrt{\frac{c-v}{c+v}}\, f_o$$

Today:

* Doppler example: redshift of our universe
* Shock waves
* Boundary conditions
  → demo with He and SF_6 in organ pipe
* Wave equation for transverse waves on a string

Doppler Example: redshift of our universe

space itself is expanding



① has moved away at $v_1 = \frac{(12d)\alpha - 12d}{t} = \frac{12d(\alpha-1)}{t}$

② has moved away at $v_2 = \frac{(10d)\alpha - 10d}{t} = \frac{10d(\alpha-1)}{t}$

⇒ velocity at which the star recedes is proportional to the distance from the star!

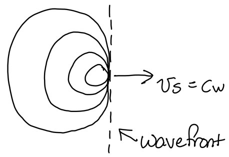
⇒ can use redshift (lowered frequency) of light from star to tell its distance!

ALSO, the farther away the star is, the longer it took for light to get here. So the greater the redshift, the farther back in time we are looking!

Shock Waves

What happens when a source moves at (or greater than) c_w?

waves pile up → SHOCK WAVE

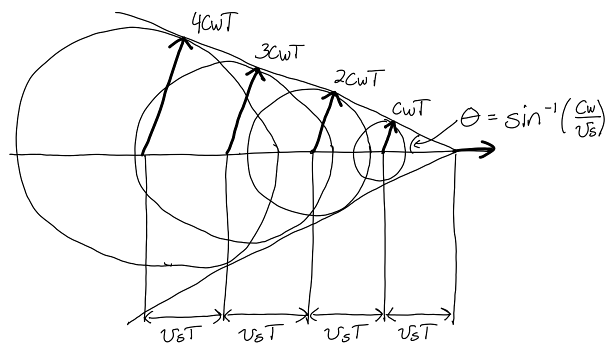


theoretically, the energy pile-up at the wavefront is ∞
→ what actually happens?
very complicated fluid dynamics problem

What useful, simple facts can we extract from this?

→ turbulence from big energy concentration → HUGE drag
⇒ very hard to break the sound barrier
(not a physics impossibility like c, just an engineering challenge)
→ once we exceed the speed of sound, the shock wave becomes a cone

MACH cone:

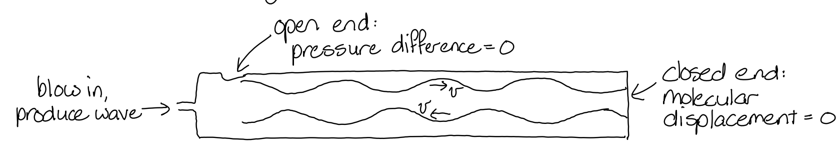


→ energy concentrates at the cone surface
→ where the cone touches the observer, the observer hears "sonic boom" = rapid delivery of LOTS of energy

Standing Waves

When a sound wave hits a hard wall, it <u>reflects</u>.
(we'll examine reflection in more quantitative detail later)

So if we introduce a sound wave into a pipe closed at one end, we have our original traveling wave away, plus a reflected traveling wave back towards.



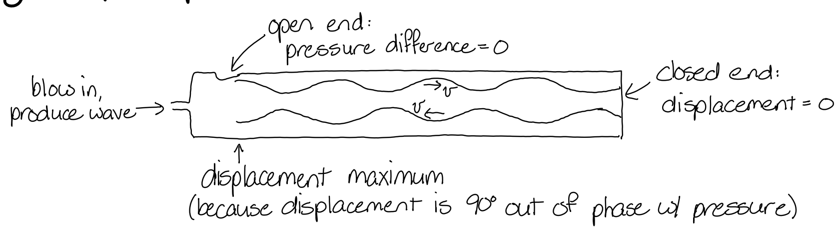
Sum of 2 opposite traveling waves (of equal amplitude):

$$A\sin(kx-\omega t) + A\sin(kx+\omega t)$$
$$= A\sin(kx)\cos(\omega t) - A\cos(kx)\sin(\omega t) + A\sin(kx)\cos(\omega t) + A\sin(kx)\cos(\omega t)$$
$$= 2A\underbrace{\sin(kx)}_{\text{has zeros = "nodes" at fixed positions in space}}\underbrace{\cos(\omega t)}_{\text{has zeros at fixed times}}$$

Can think of this as $\underbrace{(2A\cos(\omega t))}_{\text{time-varying amplitude}}\ \underbrace{\sin(kx)}_{\text{fixed wave in space = "standing wave"}}$

For a given pipe, the only waves which will really "stand" without destructive interference between → and ← must satisfy specific "boundary conditions" set by geometry of pipe.

Organ Pipes: open end + closed end:



# of wavelengths fitting in L must be $n+\frac{1}{4}$ or $n+\frac{3}{4}$

$\Rightarrow$ # of wavelengths fitting in 2L is $n+\frac{1}{2}$

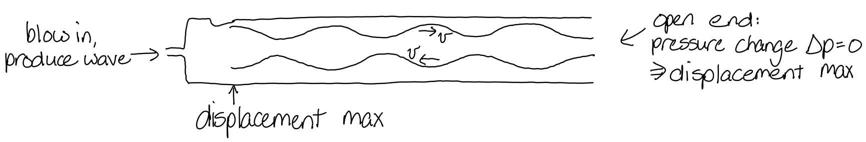
$\Rightarrow L = \left(\frac{n}{2}+\frac{1}{4}\right)\lambda \quad \Rightarrow \lambda = \frac{L}{\left(\frac{n}{2}+\frac{1}{4}\right)}$

$\Rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi\left(\frac{n}{2}+\frac{1}{4}\right)}{L}$

$\Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}(k v_{sound}) = v_{sound}\left(\frac{\frac{n}{2}+\frac{1}{4}}{L}\right)$

0^{th} harmonic: $n=0 \Rightarrow f_0 = \frac{v_{sound}}{4L}$

Organ Pipes: 2 open ends



# of wavelengths fitting in L must be $n/2$

$\Rightarrow L = \frac{n}{2}\lambda$

$\Rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi\left(\frac{n}{2}\right)}{L}$

$\Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}(k v_{sound}) = v_{sound}\left(\frac{\frac{n}{2}}{L}\right)$

1^{st} harmonic: $n=1 \Rightarrow f_1 = \frac{v_{sound}}{2L}$

Example: Organ Pipe with He vs. SF_6

Quiz: What is the ratio of the frequencies you will hear when blowing these 2 gases through an organ pipe?

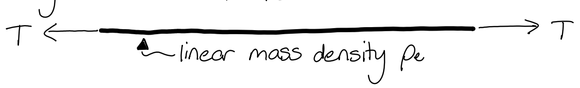
He: $M_{mol} = 4g$

SF_6: $M_{mol} = 32\,g + 6\cdot 19\,g = 146\,g$

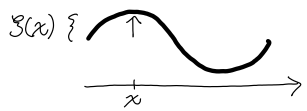
(SF_6 is used as an insulating gas in electronic components, replaces more toxic PCBs. Unfortunately it's also a greenhouse gas.)

Transverse waves on a string

string stretched with tension T

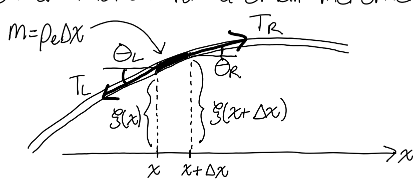


What if we vibrate the string vertically, transverse to the direction of the string?



We still refer to the displacement at x as $\xi(x)$, even though it's in an orthogonal direction now.

We'll write the equation of motion for a small increment of this string:



Assume small displacements → θ_L and θ_R are small

$$\sum F_x = -T_L\cos\theta_L + T_R\cos\theta_R = 0$$

because string is not moving left-right

$$-T_L\Big(1-\underbrace{\cancel{\tfrac{1}{2}\theta_L^2}}_{\text{small}}\Big) + T_R\Big(1-\underbrace{\cancel{\tfrac{1}{2}\theta_R^2}}_{\text{small}}\Big) = 0$$

$$\Rightarrow T_L = T_R$$

$$\sum F_y = -T_L\sin\theta_L + T_R\sin\theta_R = \underbrace{\rho_\ell\Delta x}_{\text{mass}}\underbrace{\frac{d^2\xi}{dt^2}}_{\text{acceleration}}$$

$$-T_L\theta_L + T_R\theta_R = \rho_\ell\Delta x\frac{d^2\xi}{dt^2}$$

these are equal → call them T

$$\rho_\ell\Delta x\frac{d^2\xi}{dt^2} = T(\theta_R-\theta_L) \approx T(\tan\theta_R - \tan\theta_L)$$

$$\approx T\left(\frac{d\xi(x+\Delta x)}{dx} - \frac{d\xi(x)}{dx}\right)$$

$$\rho_\ell\frac{d^2\xi}{dt^2} = T\frac{\left(\frac{d\xi(x+\Delta x)}{dx}-\frac{d\xi(x)}{dx}\right)}{\Delta x} \approx T\frac{d^2\xi}{dx^2}$$

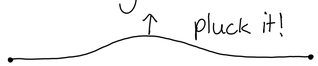
$$\boxed{\frac{\partial^2\xi}{\partial t^2} = \frac{T}{\rho_\ell}\frac{\partial^2\xi}{\partial x^2}}$$

wave equation for transverse waves on a string

Quiz:
a) what is the dispersion relation of these waves?
b) what is their velocity?
c) what is the energy transmitted by a wave $\xi_0\sin(kx-\omega t)$?

Standing waves on a string

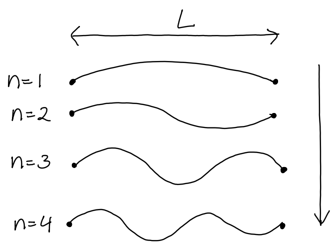
Consider a string fixed on both ends:



- → pulse will travel to either end & reflect
- → we'll soon get lots of counter-propagating wave components
- → most components will interfere destructively, but the components that satisfy the boundary conditions will survive
- → "standing wave"

$$A\sin(kx-\omega t) + A\sin(-kx-\omega t) = \underbrace{2A\sin(kx)}\cos(\omega t)$$

amplitude has "nodes" = zeros at fixed positions in space
⇒ must satisfy boundary conditions



harmonics = increasing #'s of ½-wavelengths can fit onto string

$$L = n\left(\frac{\lambda}{2}\right) = n\left(\frac{2\pi/k}{2}\right)$$

$$\Rightarrow k = \frac{n\pi}{L}$$

Dispersion relation of string tells us that

$$\omega = c_w k = \sqrt{\frac{T}{\rho_\ell}}\,k$$

$$\Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{T}{\rho_\ell}}\,k = \frac{1}{2\pi}\sqrt{\frac{T}{\rho_\ell}}\,k = \frac{1}{2\pi}\sqrt{\frac{T}{\rho_\ell}}\,\frac{n\pi}{L} = \frac{n}{2L}\sqrt{\frac{T}{\rho_\ell}}$$

"fundamental frequency" $= f_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho_\ell}}$

"harmonics" $= f_n$ for $n>1$

Quiz: If your instrument is flat (meaning f is too low), should you loosen or tighten the string?

How to excite the fundamental?

1. pluck string in middle (both ends fixed)
2. drive end at correct frequency

How to excite the harmonics?

1. enforce a node → clamp string loosely at a particular location

2. drive end at correct frequency

(note: in the lecture demo, in case ②, the driving amplitude is so small that the driven end is very nearly a node)

Musical instruments:

String instruments: use method ①. Plucking action gives a poorly controlled mix of frequencies. The instrument selects & resonates at the frequencies that satisfy the boundary conditions, resulting in constructive interference. The dominant response is the fundamental frequency. But the relative ratios of the higher harmonics are what distinguishes an A on a piano from an A on a violin.

Wind instruments: use a combination of methods ① and ②. Change the slide on the trombone → seven positions give you only seven fundamental frequencies. But can use lips to drive higher harmonics at each slide position.

Summary

Shock waves
- * accumulation of energy in wavefront or cone when source exceeds the speed of sound
- * can get light shock waves too! = "Cerenkov radiation"

Standing wave: $A\sin(kx)\cos(\omega t)$

Boundary conditions:

open+closed ends: $f_n = \dfrac{(2n+1)v_{sound}}{4L}$

2 open ends: $f_n = \dfrac{n v_{sound}}{2L}$

Transverse wave on a string: $\dfrac{\partial^2 \xi}{\partial t^2} = \dfrac{T}{\rho_\ell}\dfrac{\partial^2 \xi}{\partial x^2}$

Next time:
- * reflections
- * multi-dimensional boundary conditions

Reading for next time: H & L, chapter 6 or Morin 4.2-4.3 and 7.1 or Georgi 9.1 and 11.1-11.3, 11.6

Organ pipe demonstration:

$v_{sound} = \sqrt{\dfrac{B}{\rho}} = \sqrt{\dfrac{\gamma P}{\rho}}$ $\quad \gamma_{He} = 5/3$

$\gamma_{SF_6} = ?? = \dfrac{C_P}{C_V}$ but it seems that nobody has measured C_V ($C_P = 97\ J/mol\cdot K$)

↳ guess $7/5$ (the point is, it's a small number just larger than 1, so the $\sqrt{\gamma_{He}/\gamma_{SF_6}}$ ratio is going to be almost 1 → this will not matter much compared to $\sqrt{\rho_{SF_6}/\rho_{He}}$)

$$\Rightarrow \frac{v_{sound}(He)}{v_{sound}(SF_6)} \approx \sqrt{\frac{\rho_{SF_6}}{\rho_{He}}} = \sqrt{\frac{146g}{4g}} \approx 6$$

Since f scales like v_{sound}, frequency of He in organ pipe should be 6× higher than frequency of SF_6 in organ pipe.

Transverse waves on a string

a) Plug in the old standby: $\xi(x,t) = \xi_0 e^{i(kx-\omega t)}$

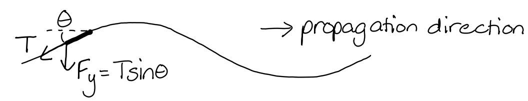
$$-\omega^2 \xi_0 e^{i(kx-\omega t)} = -k^2\left(\frac{T}{\rho_\ell}\right)\xi_0 e^{i(kx-\omega t)}$$

$$\Rightarrow \omega = \sqrt{\frac{T}{\rho_\ell}}\,k$$

b) $v_{phase} = v_{group} = \dfrac{\omega}{k} = \sqrt{\dfrac{T}{\rho_\ell}}$

↖ only equal because dispersion is linear

c) Energy transmission: take $\xi(x,t) = \xi_0 \sin(kx-\omega t)$



Power transmitted = (Force acting on a piece of string from left, which is making it move) · (Velocity of moving piece of string)

$$\langle P\rangle = \left\langle \frac{dE}{dt}\right\rangle = \left\langle (-T\sin\theta)\left(\frac{d\xi}{dt}\right)\right\rangle$$

$\sin\theta \approx \tan\theta \approx \dfrac{d\xi}{dx}$

$$\langle P\rangle = \left\langle -T\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial t}\right)\right\rangle = \langle -T(k\xi_0\cos(kx-\omega t))(-\omega\xi_0\cos(kx-\omega t))\rangle$$

$$= T\omega k\xi_0^2\langle\cos^2(kx-\omega t)\rangle = \frac{1}{2}T\omega k\xi_0^2$$

↖ $k = \dfrac{\omega}{v_{phase}} = \omega\sqrt{\dfrac{\rho_\ell}{T}}$

$\boxed{\langle P\rangle = \frac{1}{2}\sqrt{\rho_\ell T}\,\omega^2\xi_0^2}$ power transmitted

Note: $\sqrt{\rho_\ell T}$ is "impedance" of string

Flat instrument: tighten your string

↖ frequency too low