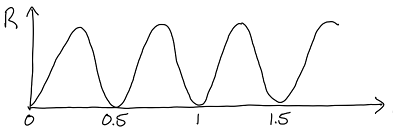
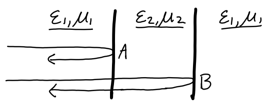
Physics 15c (Hoffman)
Lecture #20
Thurs, Nov 18, 2010

Diffraction, Rayleigh criterion

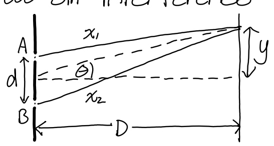
Last time:

* Thin film interference:





* Two slit interference

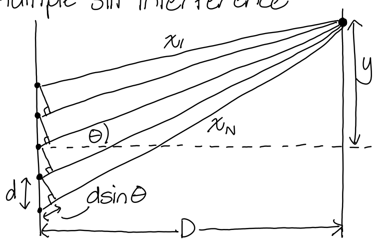


Projection onto screen:

$$I = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \approx I_0 \cos^2\left(\frac{\pi d y}{\lambda D}\right)$$

for small θ, i.e. D >> y

* Multiple slit interference



Projection onto screen:

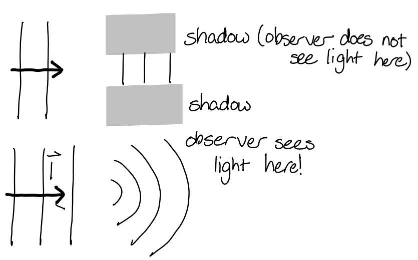
$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin(Nx)}{N\sin x}\right)^2$$

where $x \equiv \frac{1}{2} k d \sin\theta$

Peaks at $\sin\theta = 0, \lambda/d, 2\lambda/d, \ldots$

→ get narrower as N increases

* Diffraction of wide slit



$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin x}{x}\right)^2$$

where $x \equiv \frac{ka\sin\theta}{2} = \frac{\pi a \sin\theta}{\lambda}$

⇒ we see a broad central maximum whose angular width is inversely proportional to the slit width

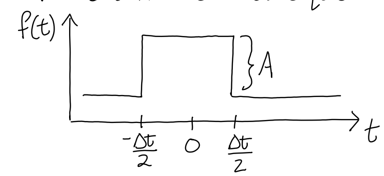
Today:

* Fourier transforms - back again!
* Interference vs. diffraction
* Airy disc, diffraction limit, Rayleigh criterion

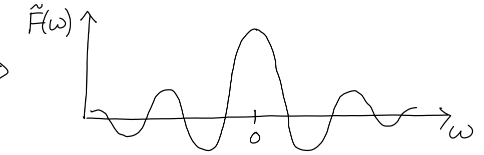
Fourier transforms

Goal: mathematical understanding for why wide slit produces narrow beam but narrow slit produces wide beam → look to FT's for analogy

Remember way back to pset #4, you calculated the Fourier transform of a square pulse:



⟺



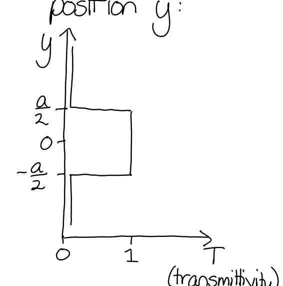
$$\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = A\Delta t \left(\frac{\sin\left(\frac{\omega\Delta t}{2}\right)}{\frac{\omega \Delta t}{2}}\right) = A\,\Delta t\, \mathrm{sinc}\left(\frac{\omega\Delta t}{2}\right)$$

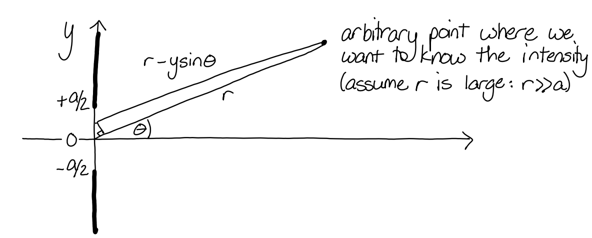
⇒ Fourier transform of a square wave is a sinc.

Looks like the diffraction pattern is the Fourier transform of the slit???

Does this mathematical connection have a physical explanation?

Yes, represent the slit as a transmittivity, as a function of position y:





At point r, the sum of all waves from the whole y-axis is:

$$E = \int_{-\infty}^{\infty} dy\, E_0\, T(y)\, e^{i[k(r - y\sin\theta) - \omega t]}$$

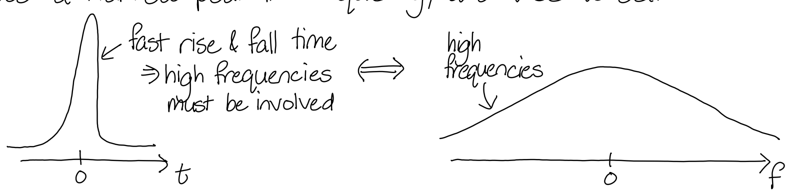
$$= E_0 e^{i(kr - \omega t)} \underbrace{\int_{-\infty}^{\infty} T(y)\, e^{iky\sin\theta}\, dy}_{\text{looks like a Fourier transform between variables } y \longleftrightarrow k\sin\theta}$$

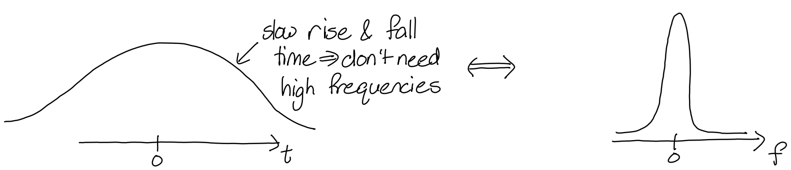
$$E = E_0 e^{i(kr-\omega t)} \tilde{F}(k\sin\theta)$$

$f(y) \Longleftrightarrow \tilde{F}(k\sin\theta)$ are Fourier pairs

Remember the Fourier transform of the Gaussian?

You reasoned that the FT of a wide peak in time gives a narrow peak in frequency, and vice versa.





So the moral of the story is that the intensity on the far side of a slit is like a Fourier transform of the shape of the slit. A wide slit gives a narrow intensity pattern but a narrow slit gives a wide intensity pattern.

Definitions:

What is the difference between **interference** and **diffraction**?

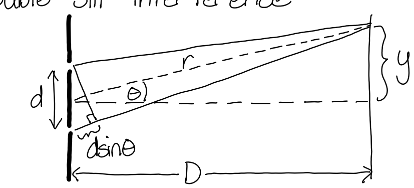
Nothing, really. Diffraction is just a limiting case of interference, for a continuous distribution of point sources.

Fraunhofer Diffraction: far-field limit
= everything we have been doing so far

Fresnel Diffraction: near-field limit
→ can no longer make r-is-large approximation
→ must take into account $1/r$ amplitude fall-off from diff. pts.
⇒ messy, no algebraic solutions ⇒ compute numerically

Approximations:

Double-slit interference



approximations:

① $r \gg d$

actual path difference $= \sqrt{D^2 + (y + d/2)^2} - \sqrt{D^2 + (y - d/2)^2}$

$$= (D^2 + y^2)^{1/2} \left\{ \sqrt{1 + \frac{d^2/4 + yd}{D^2 + y^2}} - \sqrt{1 + \frac{d^2/4 - yd}{D^2 + y^2}} \right\}$$

$$= r \left\{ \sqrt{1 + \frac{d}{r}\left(\frac{d}{4r} + \frac{y}{r}\right)} - \sqrt{1 + \frac{d}{r}\left(\frac{d}{4r} - \frac{y}{r}\right)} \right\}$$

small by assumption $d \ll r$; $\sin\theta < 1$

$$\approx r \left\{ 1 + \frac{1}{2}\frac{d}{r^2}\left(\frac{d}{4} + y\right) - 1 - \frac{1}{2}\frac{d}{r^2}\left(\frac{d}{4} - y\right) \right\}$$

$$= r \left\{ \frac{dy}{r^2} \right\} = d\,\frac{y}{r} = d\sin\theta$$

(note: pretty good as long as $r > 5d$)

What about the fractional difference between $|E_A|$ and $|E_B|$?

$$\frac{|E_A|-|E_B|}{\frac{1}{2}(|E_A|+|E_B|)} \propto \frac{\frac{1}{\sqrt{D^2+(y-d/2)^2}} - \frac{1}{\sqrt{D^2+(y+d/2)^2}}}{\frac{1}{2}\left(\frac{1}{\sqrt{D^2+(y-d/2)^2}} + \frac{1}{\sqrt{D^2+(y+d/2)^2}}\right)}$$

$$\frac{\sqrt{D^2+(y+d/2)^2} - \sqrt{D^2+(y-d/2)^2}}{\frac{1}{2}\left(\sqrt{D^2+(y+d/2)^2} + \sqrt{D^2+(y-d/2)^2}\right)}$$

We have already computed the numerator to 1st order $= d\sin\theta$

Denominator $= \frac{D}{2}\left\{\sqrt{1+\left(\frac{y}{D}+\frac{d}{2D}\right)^2} + \sqrt{1+\left(\frac{y}{D}-\frac{d}{2D}\right)^2}\right\} \geq D$

$\Rightarrow$ fractional difference between $|E_A|$ and $|E_B|$ is only

$$\frac{d\sin\theta}{D} \ll 1$$

In comparison, if $d\sin\theta$ has a value comparable to or greater than λ, then E_A and E_B may differ by as much as 100% due to their relative phase shift.

$\Rightarrow$ From the $r \gg d$ approximation, we conclude

$$I = I_0\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$$

② small θ (in other words $y \ll D$)

$$\Rightarrow I = I_0\cos^2\left(\frac{\pi dy}{\lambda r}\right) \approx I_0\cos^2\left(\frac{\pi dy}{\lambda D}\right)$$

(because $\sin\theta \approx \tan\theta$ for small θ)

(note: pretty good for $\theta < 10°$)

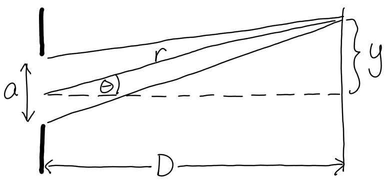
③ any other hidden assumptions/approximations?

a) we're assuming that our light is monochromatic
  → all the same wavelength

b) we're also assuming that the slits are point sources
  → no interference btwn light from different parts of the same slit
  (note: pretty good as long as diameter of slit $\ll \lambda$)

c) we're also assuming light from the 2 slits is coherent
  → interference pattern has maximum at center and is stable in time
  (note: satisfied if light is produced by the same single coherent source behind the slits)

Single wide slit interference



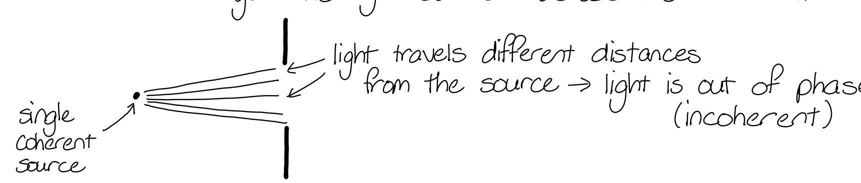
approximations:

① still need $r \gg a$
  → can use path difference approximation
  → can approximate field magnitude the same from all parts of the slit

② small θ?
  → doesn't actually help us much
  we can write the intensity analytically in terms of $\sin\theta$ without making any small-θ approximations

③a monochromatic - means that interference patterns won't split up by color

③b wide slit - how wide?
  can calculate analytically for any ratio of a/λ
  practical limitation comes from coherence
  if slit is too wide then it's very hard to get the light coherent across the whole slit



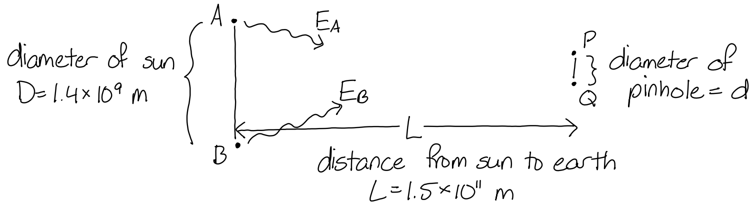
→ this gives a rough criterion on a vs. λ depending on how far away the source is

How to use the sun as a coherent source?

→ send sunlight through a pinhole

Consider 2 pts on opposite sides of the sun:

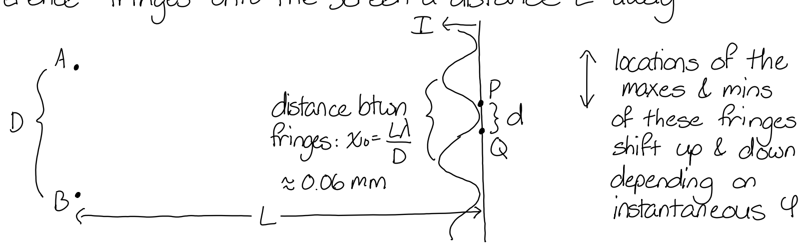
And consider the pinhole itself as the "screen"



Consider the fields from A and B, E_A and E_B, as they reach the pinhole. Think of the pinhole itself as the whole "screen". We care only about the light that gets through the pinhole, so we care only about the sun's field impinging within the d of the pinhole.

A is a point source and B is a point source. Their relative phase may fluctuate in time b/c the atoms oscillating and radiating at A are too far away to be influenced by or lock phase with the atoms oscillating and radiating at B. But during any given instant Δt, there is a well defined phase difference φ between A and B. [Note: $\Delta t < (\Delta f)^{-1}$ where Δf is the frequency spread or bandwidth of the solar spectrum.]

So, during that instant, A and B will project a pattern of interference fringes onto the screen a distance L away:



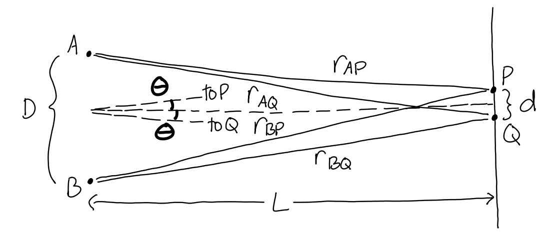
The locations of the maxima & minima of the fringes shift up & down on a time scale $\sim(\Delta f)^{-1}$ depending on the instantaneous value of φ, the relative phase between A & B.

But the distance btwn successive maxima is fixed at

$$x_0 = \frac{L\lambda}{D} \approx \frac{(1.5\times10^{11}\,m)(5.5\times10^{-7}\,m)}{(1.4\times10^{9}\,m)} = 0.06\ mm$$

no matter what the relative phase φ.

$\vec{E}$-field at "screen" (=pinhole)



$$E_P = E_0\left(e^{ikr_{AP}} + e^{ikr_{BP}+i\varphi}\right)e^{-i\omega t}$$

factor out $e^{i(kr_{AP}+kr_{BP}+\varphi)/2}$

$$= E_0\left(e^{i(kr_{AP}-kr_{BP}-\varphi)/2} + e^{i(-kr_{AP}+kr_{BP}+\varphi)/2}\right)e^{i(kr_{AP}+kr_{BP}+\varphi)/2}e^{-i\omega t}$$

$$= \underbrace{2E_0\cos\left(\frac{k(r_{AP}-r_{BP})-\varphi}{2}\right)}_{\text{space-oscillating part}}\underbrace{e^{i\left[\frac{k(r_{AP}+r_{BP})}{2}+\frac{\varphi}{2}-\omega t\right]}}_{\text{time-oscillating part}}$$

space-oscillating part undergoes significant phase shift between P and Q only when

$\frac{k(r_{AP}-r_{BP})-\varphi}{2}$ changes by $\sim\frac{1}{10}$ of a cycle $= \frac{2\pi}{10}$

from $\frac{k(r_{AQ}-r_{BQ})-\varphi}{2}$

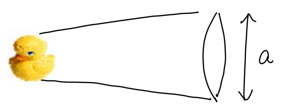
$$\frac{k(r_{AP}-r_{BP})-\varphi}{2} - \frac{k(r_{AQ}-r_{BQ})-\varphi}{2}$$

$$= \frac{k(-D\sin\theta)-\varphi}{2} - \frac{k(D\sin\theta)-\varphi}{2}$$

Optical devices

We use lenses to collect light from objects
(e.g. telescopes, microscopes, cameras, human eyes)

Lenses have finite aperture

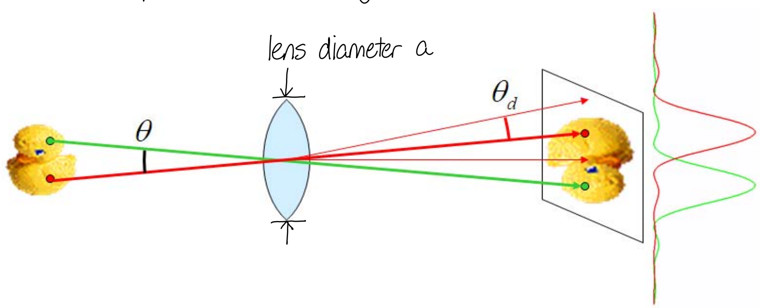


$\Rightarrow$ it's like passing light through a hole $\Rightarrow$ diffraction

Light spreads out according to $\sin\theta = \frac{\lambda}{a}$ $\Rightarrow$ this blurs the image

Optical resolution

Consider 2 points of the object:



Light from these 2 points ends up at 2 points, but each point in the image is spread out due to diffraction by the lens (like a wide slit of width a).

θ_d is the angle of blurring in the image: $\sin\theta_d = \frac{\lambda}{a}$

If the angle θ between 2 points is $\theta < \theta_d$, the points cannot be distinguished (or "resolved") in the image.

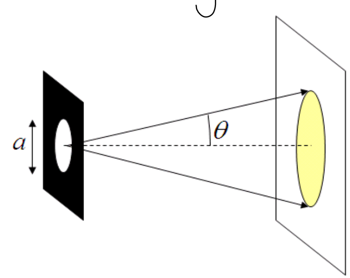
We can consider θ_d as the ultimate resolution
→ it's fundamental, due to the wave nature of light
→ it's determined by the "aperture" (= diameter) of the lens $\Rightarrow$ bigger lenses are always better

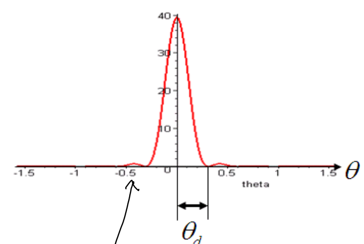
Note: We calculated the Fraunhofer diffraction for a slit by performing a 1-dimensional integral:

$$E = \int_{-\infty}^{\infty} dy\, E_0 \underbrace{T(y)}_{} e^{i[k(r - y\sin\theta) - \omega t]}$$

this is the transmissibility of the slit in one dimension
(assume it is "invarient" or infinitely wide in the other direction)

Now we need to do a 2-dim integral in polar coords for a circular lens. The solution to this integral is called an "Airy" function:

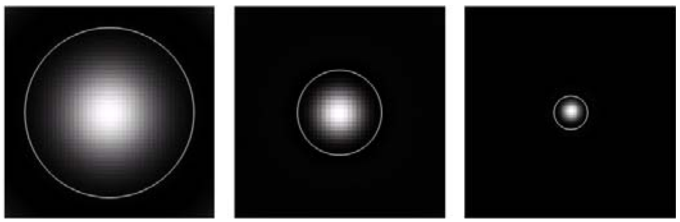




Note: there are fringes too but these are usually negligible compared to main peak.

Airy Disc: actual 2-dim shape of the light pattern

$a=\lambda$ $\quad$ $a=2\lambda$ $\quad$ $a=5\lambda$



There should be fringes, but they're too faint to see.

Diffraction limit or Rayleigh criterion

Resolution of a 2-dim optical device with aperture a is:

$$\theta \approx \sin\theta > 1.22\frac{\lambda}{a}$$

Note: $\sin\theta \approx \frac{d}{r}$

so if we make d smaller or r bigger, then we run into trouble

Quiz: what is the smallest distance between 2 lines that your eye can resolve?

Hint: you will need to estimate several quantities

(a) wavelength of optical light: $\lambda \sim$

(b) diameter of your pupil: $a \sim$

(c) smallest distance from your eye at which you can focus on an object: $D \sim$
(Note: this is simply a physiological limit imposed by your eye muscles, and it actually gets longer as you age and your eye muscles get less flexible)

Follow-up: can you resolve blue or red light better?

Quiz Answers

(a) $\lambda \sim 550$ nm
(b) $a \sim 5$ mm
(c) $D \sim 10$ cm

d = distance between 2 objects we are trying to resolve

$$\theta \approx \sin\theta \approx \frac{d}{D} \quad \text{and} \quad \sin\theta > 1.22\frac{\lambda}{a}$$

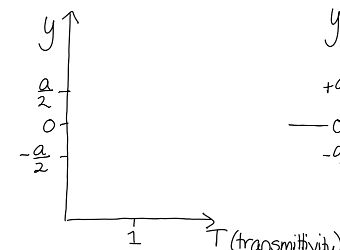
$$\Rightarrow \frac{d}{D} > 1.22\frac{\lambda}{a}$$

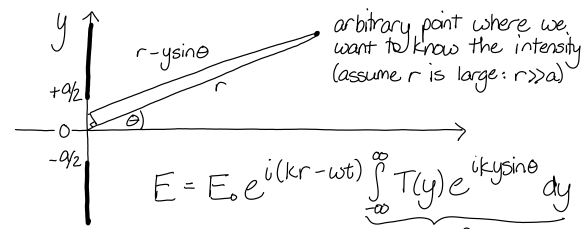
$$d > 1.22\frac{\lambda D}{a} = \frac{1.22(5.5\times10^{-7})(0.1)}{0.005} = 0.013\text{ mm} = 13\,\mu\text{m}$$

~width of the finest of human hairs

Summary

Fourier Transforms:



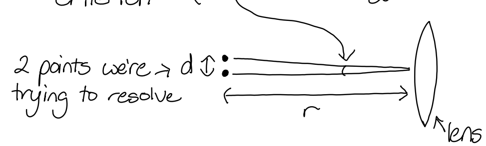


$$E = E_0 e^{i(kr-\omega t)} \underbrace{\int_{-\infty}^{\infty} T(y) e^{iky\sin\theta}\, dy}_{\text{Fourier transform between variables } y \longleftrightarrow k\sin\theta}$$

Optical resolution

The resolution of an optical device with aperture a is:

"Rayleigh criterion" $\left\{ \sin\theta > 1.22\frac{\lambda}{a} \right.$



Note: $\sin\theta \approx \frac{d}{r}$

so if we make d smaller or r bigger, then we run into trouble

Interference vs. Diffraction → what is the difference?

Nothing; diffraction is just the limiting case of interference for a continuous distribution of point sources (i.e. wide slit)