Physics 15c (Hoffman)
Lecture # 6
Tues, Sept 21, 2010

Beats & Dispersion Relations

Last time:

* continuum limit for 
* wave equation for **longitudinal waves**

$$\underbrace{\rho_e}_{\text{linear mass density}} \frac{\partial^2}{\partial t^2} \underbrace{\xi}_{\text{displacement from equilibrium}}(\underbrace{x}_{\text{equilibrium position}}, t) = \underbrace{E}_{\text{elastic modulus}} \frac{\partial^2}{\partial x^2} \xi(x,t)$$

* recalled that finite system of N masses has **normal modes**

$$\xi^{(m)} = \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}}_{\text{amplitude ratio}} \underbrace{e^{i\omega_m t}}_{\text{oscillation at fixed frequency}}$$

* used this tip from finite N systems to guess a soln to continuous eqn:

$$\xi(x,t) = a(x) e^{i\omega t}$$

$\to$ plug in $\to$ find: $\xi(x,t) = A e^{i(\underbrace{kx}_{\text{space}} \pm \underbrace{\omega t}_{\text{time}})}$

Again, we have normal modes, which are separable into the product of a space-dependent part and a time-dependent part.
But here we have continuous (infinite) degrees of freedom.
- $\Rightarrow$ ω can take any value!
- $\to$ but for each ω, there is a specific space-dependent part
- $\to$ value of k is constrained by ω
  (actually, get 2 values of k (+ and −) for each ω)
- $\to$ relationship between k & ω is called "dispersion relation"

$$\underbrace{\omega}_{\text{angular frequency}} = \frac{2\pi}{\underbrace{T}_{\text{period}}} \qquad \underbrace{k}_{\text{wavenumber}} = \frac{2\pi}{\underbrace{\lambda}_{\text{wavelength}}}$$

* **phase velocity**: how much does the phase advance per unit time?
  one wavelength per period:

$$v_{phase} = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$$

* **dispersion relation**:
  We found this by plugging our solution $\xi(x,t) = Ae^{i(kx-\omega t)}$ into the wave equation:

$$\rho_e \frac{\partial^2}{\partial t^2} \xi(x,t) = E \frac{\partial^2}{\partial x^2} \xi(x,t)$$

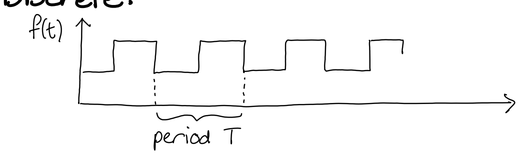
$$\rho_e(-\omega^2)\xi(x,t) = E(-k^2)\xi(x,t)$$

$$\Rightarrow \underbrace{k = \pm\omega\sqrt{\frac{\rho_e}{E}}}$$

This particular dispersion relation is linear!
Means that $v_{phase} = \frac{\omega}{k} = \sqrt{\frac{E}{\rho_e}}$ is same for all normal modes!

* **Fourier transforms**

① Discrete:



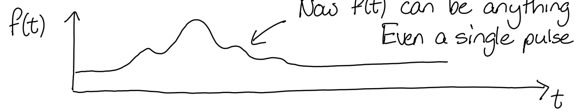
The discrete (but infinite) set of functions $\cos\left(\frac{2\pi n t}{T}\right)$ and $\sin\left(\frac{2\pi n t}{T}\right)$ form a <u>complete</u> basis for the set of all periodic functions $f(t)$ with period T.
$\Rightarrow$ We can write any periodic $f(t)$ with period T as:

$$f(t) = \underbrace{a_0}_{\text{constant offset}} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right\}$$

where a_n and b_n come from the "inner product" of $f(t)$ with the basis state, e.g.

$$a_n = \frac{1}{T} \int_0^T f(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$

② Continuous:



Now f(t) can be anything at all, periodic or not. Even a single pulse!

The continuous, infinite set of functions $e^{i\omega t}$ (where ω runs over all real numbers from $-\infty$ to $+\infty$) form a complete basis for the set of all functions $f(t)$.

well... all well-behaved functions, e.g. finite # of finite discontinuities, etc

$$f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega t} d\omega$$

basis vectors

coefficient of linear combo of basis vectors

need to integrate instead of add to find linear combo, because ω is continuous

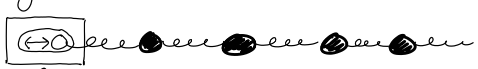
Now $\tilde{F}(\omega)$ comes from an "inner product" of $f(t)$ with basis state:

$$\tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

[Note: inner product is a measure of the "overlap" of 2 functions:

$$\int_{-\infty}^{\infty} f(t) g^*(t) dt$$
]

* exciting a wave:



driving motor:

$= \xi(t, x=0)$

→ can be decomposed into $\int e^{-i\omega t}$

→ each component moves at the same

$$v_{phase} = \frac{\omega}{k} = \sqrt{\frac{E}{\rho_\ell}}$$

⇒ whole

moves unchanged at $v_{phase} = \sqrt{E/\rho_\ell}$

⇒ Any function that moves at $v_{phase} = \sqrt{E/\rho_\ell}$ satisfies our wave equation! $\xi(x,t) = f(x - v_{phase} t)$

Seems pretty stupid → why did we bother?

Anything that travels with the correct velocity is a wave...

But wait! We have chosen a system which satisfies 2 specific conditions:

① linearity of forces
  → tells us that solutions add
② linearity of dispersion
  → tells us that all solutions travel at same speed

These are 2 different conditions!
We will find many systems which satisfy ① but not ②.

Goals For Today

* Beats
* v_{phase} vs. v_{group}
* Dispersion relations

Let's do one more concrete example of linear dispersion.
Look at the superposition of 2 different normal modes at ω_1, ω_2.

Each mode is $\sin(kx - \omega t)$ with $\frac{\omega}{k} = \sqrt{\frac{E}{\rho_\ell}} = 1$

but take ω_2 to be 10% bigger than ω_1:

$\omega_1 = 2.0 \qquad k_1 = 2.0$

$\omega_2 = 2.2 \qquad k_2 = 2.2$

→ get beats!
→ but both sine waves propagate at same speed
→ beats also propagate at same speed

} in this special case of linear dispersion
  → fixed v_{phase}

Let's do a more formal derivation of the beat velocity:

$$\xi(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$

$\downarrow$ trig formulas

$$= 2\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$$

$$= \underbrace{2\cos(\bar{k}x - \bar{\omega}t)}_{\text{carrier wave with average wavelength \& average frequency}}\underbrace{\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)}_{\text{shape of the beats}}$$

Think of your AM radio station:

This is the frequency you tune your radio to

But this is the part that actually contains the information

So how fast does the information travel?

→ NOT at the phase velocity of the carrier signal $\frac{\bar{\omega}}{\bar{k}}$

→ travels at the velocity of the beats $\frac{\Delta\omega}{\Delta k} \approx \frac{d\omega}{dk} \equiv v_{group}$

In this particular simple example 

the dispersion relation is linear, so it does happen to be true that:

$$v_{group} = \frac{d\omega}{dk} = \sqrt{\frac{dE}{d\rho_e}} = \frac{\omega}{k} = v_{phase}$$

BUT, - not always true!

Note: special relativity says that information can never travel faster than the speed of light: $c = 3\times10^8$ m/s

"Nothing travels faster than the speed of light, with the possible exception of bad news, which obeys its own set of laws." - Douglas Adams

But we can have $v_{phase} > c$ because the pure sine wave carries no information - it just always was & always will be ⇒ no news is transmitted.

$v_{group} \le c$ except in some very contrived examples...

But velocity of information is <u>always</u> $\le c$

Let's look at an example of a system with a nonlinear dispersion relation.

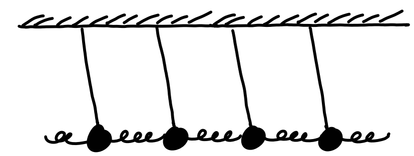
(Jargon alert! Often people say that a system with linear dispersion relation is "dispersionless" while a system with nonlinear dispersion relation "has dispersion". All systems have some equation which relates $\omega(k)$ - and this relation is always called the dispersion relation. But according to the jargon, some systems have dispersion and some don't.)

Earlier we had:  ⇒ no dispersion

$$m\frac{d^2}{dt^2}\xi_n = -k_s(\xi_n - \xi_{n-1}) - k_s(\xi_n - \xi_{n+1})$$

add subscript for spring (to distinguish from wave number)

Now we consider:



New equation of motion:

$$m\frac{d^2\xi}{dt^2} = -k_s(\xi_n - \xi_{n-1}) - k_s(\xi_n - \xi_{n+1}) - \frac{mg}{L}\xi_n$$

<u>**Quiz**</u> Take the continuum limit to find the wave equation for this system. (don't turn the page! think about it & talk with neighbors.)

Then go ahead and find a solution to this wave equation.

$$m\frac{d^2\xi_n}{dt^2} = -k_s(\xi_n-\xi_{n-1}) - k_s(\xi_n-\xi_{n+1}) - \frac{mg}{L}\xi_n$$

relabel: $\xi_n \to \xi(x,t)$

$\xi_{n\pm1} \to \xi(x\pm\Delta x, t)$

$$m\frac{\partial^2\xi}{\partial t^2} = (\Delta x)^2 k_s \left\{ \frac{\frac{\xi(x+\Delta x)-\xi(x)}{\Delta x} - \frac{\xi(x)-\xi(x-\Delta x)}{\Delta x}}{\Delta x} \right\}$$

$$= (\Delta x)^2 k_s \frac{\partial^2\xi}{\partial x^2} - \frac{mg}{L}\xi$$

$$\frac{m}{\Delta x}\frac{\partial^2\xi}{\partial t^2} = \Delta x\, k_s \frac{\partial^2\xi}{\partial x^2} - \frac{m}{\Delta x}\frac{g}{L}\xi$$

$$\rho_e\frac{\partial^2\xi}{\partial t^2} = E\frac{\partial^2\xi}{\partial x^2} - \rho_e\omega_0^2\xi$$

natural frequency of a single pendulum: $\omega_0 = \sqrt{\frac{g}{L}}$

$$\frac{\partial^2\xi}{\partial t^2} = \frac{E}{\rho_e}\frac{\partial^2\xi}{\partial x^2} - \omega_0^2\xi$$

OK, let's guess a solution: $\xi(x,t) = a(x)e^{i\omega t}$

Remember, this is a "normal mode" which separates into space and time parts. Every ω should determine a specific $a(x)$.

Plug in:

$$-\omega^2 a(x)e^{i\omega t} = \frac{E}{\rho_e}e^{i\omega t}\frac{d^2a(x)}{dx^2} - \omega_0^2 a(x)e^{i\omega t}$$

$$\Rightarrow \underbrace{\frac{d^2a(x)}{dx^2} = -\left(\frac{\omega^2-\omega_0^2}{E/\rho_e}\right)a(x)}$$

This looks like a simple harmonic oscillator if $\omega > \omega_0$

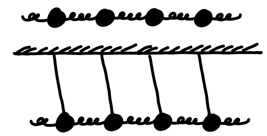
$\Rightarrow$ we can write the solution as

$\xi(x,t) = Ae^{i(kx\pm\omega t)}$ but with $k = \sqrt{\frac{\omega^2-\omega_0^2}{E/\rho_e}}$

AH-HAH! this is <u>not</u> a linear dispersion.

OK, so the normal modes still look like waves, as we expect:

$\xi(x,t) = Ae^{i(kx\pm\omega t)}$ for both and



but the relation between k and ω has changed.

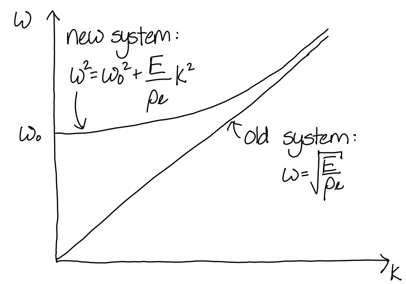
To calculate the propagation velocity, we look at:

$$v_{phase} = \frac{\lambda}{T} = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\omega^2-\omega_0^2}{E/\rho_e}}} = \sqrt{\frac{E}{\rho_e}}\frac{\omega}{\sqrt{\omega^2-\omega_0^2}}$$

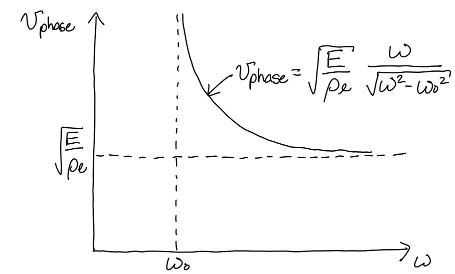
phase velocity depends on frequency!

Different sine waves propagate at different frequencies!

dispersion relation



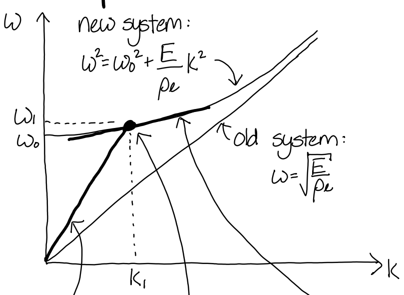
phase velocity



* dispersive waves have no propagating solutions for $\omega < \omega_0$
  → low frequency cutoff at $\omega = \omega_0$
* phase velocity goes to infinity at cutoff
  → is this unphysical? violates relativity?
  → no, v_{phase} does not represent information transfer

Compare phase and group velocities:

dispersion relation

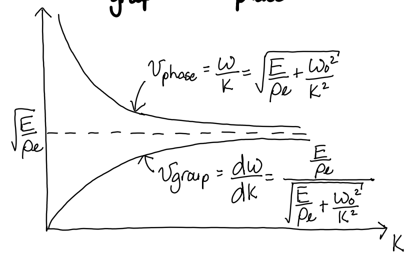


compare phase & group velocities @ arbitrary point k_1, ω_1

phase velocity $= \frac{\omega}{k}$
= slope of line to the point being considered

group velocity $= \frac{d\omega}{dk}$
= slope of line tangent at the point being considered

v_{group} vs. v_{phase}



v_{group} remains less than v_{phase}
→ information never travels faster than c

Waves can't exist below the cutoff frequency ω_0
→ exactly what is going on there?

① First look at ω exactly equal ω_0

wave eqn: $\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{E}{\rho_e} \frac{\partial^2 \xi(x,t)}{\partial x^2} - \omega_0^2 \xi(x,t)$

plug in: $\xi(x,t) = a(x) e^{i\omega_0 t}$

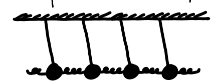
$-\omega_0^2 a(x) e^{i\omega_0 t} = \frac{E}{\rho_e} \frac{d^2 a(x)}{dx^2} e^{i\omega_0 t} - \omega_0^2 a(x) e^{i\omega_0 t}$

$\Rightarrow \frac{d^2 a(x)}{dx^2} \underbrace{e^{i\omega_0 t}}_{\text{this is not zero (must have magnitude 1)}} = 0$

$\Rightarrow \frac{d^2 a(x)}{dx^2} = 0$

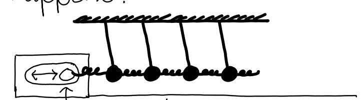
⇒ solution $a(x) = A + Bx$

→ but we can't have solutions go to ∞ at $x \to \pm\infty$
→ boundary conditions tell us that $B = 0$
(boundary conditions are the space version of initial conditions: they set the constant multipliers in front of the normal modes 1 and x)
→ this leaves us with $\xi(x,t) = Ae^{i\omega_0 t}$
no position dependence!



all swing in synch!

② What about $\omega < \omega_0$?
→ waves can't propagate at these low frequencies
→ but there's nothing to stop us from attaching a motor and trying to drive at these frequencies
→ what happens?



motor drives:
$\xi(0,t) = Ae^{-i\omega t}$ where $\omega < \omega_0$

As usual, we write the solution: $\xi(x,t) = Ae^{i(kx - \omega t)}$

but the wave eqn gives us imaginary k!

$$k = \pm \frac{\sqrt{\omega^2 - \omega_0^2}}{\sqrt{E/\rho_e}} = \pm i \frac{\sqrt{\omega_0^2 - \omega^2}}{\sqrt{E/\rho_e}}$$

how does this make physical sense?

For an imaginary k, we define: $k = \pm i\Gamma$, for $\Gamma > 0$

$\left(\text{in this case, } \Gamma = \frac{\sqrt{\omega_0^2 - \omega^2}}{\sqrt{E/\rho_e}} \right)$

The solution becomes

$$\xi(x,t) = Ae^{i(kx - \omega t)} = Ae^{i(i\Gamma x - \omega t)}$$
$$= Ae^{\mp \Gamma x} e^{-i\omega t}$$

Again consider boundary conditions: $e^{+\Gamma x}$ goes to ∞

So we must pick $e^{-\Gamma x}$

$$\Rightarrow \xi(x,t) = A e^{-\Gamma x} e^{-i\omega t}$$

i.e. the solution shrinks exponentially with x

→ your waves never go much farther than $1/\Gamma$

OK, so we have covered all bases.

1. $\omega > \omega_0$ → traveling waves described by dispersion relation
2. $\omega = \omega_0$ → uniform (coherent) oscillation over entire space
3. $\omega < \omega_0$ → exponentially attenuating with distance

Summary

* studied continuous waves
* dispersion relation $\omega(k)$
  obtained by plugging trial solution into wave equation
* any function can be represented by sum or integral of waves
  → any function can be a wave as long as it propagates with the right velocity
* "dispersion-less": linear dispersion relation
  → all modes propagate w/ same phase velocity
  → pulse maintains its shape
* "dispersion": non-linear dispersion relation
  → modes propagate with different phase velocity
  → pulse distorts over time
* information travels at group velocity

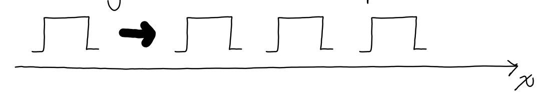
$$v_{phase} = \frac{\omega}{k} \qquad v_{group} = \frac{d\omega}{dk}$$

Next time:

Imagine a pulse being sent over a distance

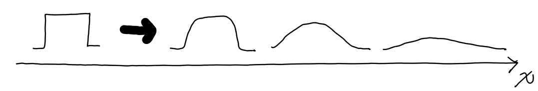
non-dispersive medium: the pulse shape is unchanged

* because all normal modes (all the sines and cosines that add together to make the pulse) have the same v_{phase}



dispersive medium: pulse shape must change → pulse distorts

* because all normal modes (all the sines and cosines that add together to make the pulse) travel at different velocities, so the pulse breaks apart



→ dispersion makes a poor medium for communication

We will investigate this in more mathematical detail next time.