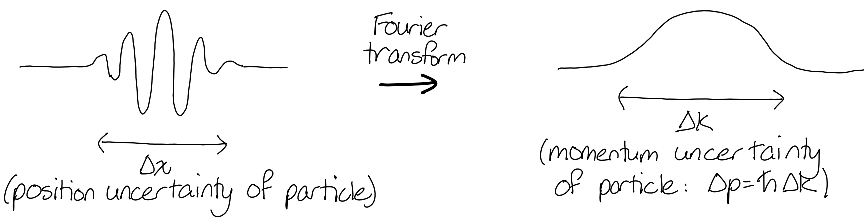
Questions:

- What do you mean by a single, finite wave packet – is it physical?

Yes, every particle is a single, finite wave packet.

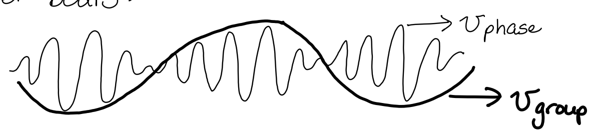


- Are sound waves dispersive?

Well... it's sort of a confusing way to phrase the question. The medium (air) through which sound waves travel is dispersive. (But not very dispersive, so you can hear each other pretty well.)

- Why does $v_{group}=d\omega/dk$ and $v_{phase}=\omega/k$?

Hand-waving: suppose we add just 2 cosines together, we get beats:



$$\cos(k_1 x-\omega_1 t)+\cos(k_2 x-\omega_2 t)$$

$$=2\cos\left(\frac{k_1+k_2}{2}x-\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x-\frac{\omega_1-\omega_2}{2}t\right)$$

$$=\underbrace{2\cos(\bar{k}x-\bar{\omega}t)}_{\text{carrier}}\underbrace{\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)}_{\text{envelope}}$$

$$\Rightarrow v_{phase}=\frac{\bar{\omega}}{\bar{k}} \qquad \Rightarrow v_{group}=\frac{\Delta\omega/2}{\Delta k/2}=\frac{\Delta\omega}{\Delta k}\approx\frac{d\omega}{dk}$$

Rigorous: see full derivation of $v_{group}=\frac{d\omega}{dk}$ in lecture #7 starting on page 7.

- Do modes describe velocity? Acceleration? What's the difference between (1,1,1) mode and a situation where all three masses are moving at the same velocity = 1?

Normal modes describe the ratio of position amplitudes. Because a normal mode describes a pattern of motion in which every mass oscillates at the same frequency, when you take a derivative of

$$x(t)=e^{i\omega t}\begin{pmatrix}x_1\\x_2\\\vdots\\x_N\end{pmatrix} \quad \text{to get} \quad v(t)=i\omega e^{i\omega t}\begin{pmatrix}x_1\\x_2\\\vdots\\x_N\end{pmatrix}$$

every term in the vector brings down the same factor of ω from the exponent, so the normal mode also describes the ratio of velocities (accelerations, jerks, etc.)

Therefore, a (1,1,1) mode does describe a situation where all 3 masses are moving at the same velocity (not necessarily velocity = 1, though.)

- Why are the eigenvectors the same for S & K but not the eigenvalues?

Suppose K and S commute (i.e. $SK=KS$).
Let v be an eigenvector of S with eigenvalue c (i.e. $Sv=cv$).

Then $SKv=KSv=K(cv)=cKv$.

$$S(Kv)=c(Kv)$$

So we showed that Kv is also an eigenvector of S with eigenvalue c. Now if S has N distinct eigenvalues, we know that Kv must be the same eigenvector of S as v was. But eigenvectors are only defined up to a constant. So the best we can do is to say

$$Kv=dv$$

d ← some constant, not necessarily c

- Why if v is an eigenvector, is its conjugate an eigenvector too?
- Why is it valid to use (v+v*)/2 and (v-v*)/2 and expect them to be eigenvectors too? (i.e. why are linear combos of eigenvectors also eigenvectors?)

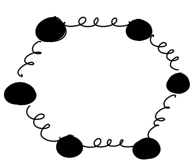
Let me put these mathematical questions back into physical context...

Question: how do we get from the mathematical imaginary solutions to the real physical solutions? and why do we need imaginary solutions at all - what do they mean, anyhow?

Answer: Imaginary solutions are a useful computational tool, which can dramatically reduce the number of pages of algebra to solve a problem. But they're only an intermediate step between the real statement of the physical problem and the real solution to the physical problem. So they're only effective if we know how to get from the intermediate imaginary step back to the real solutions at the end.

Note: I can't really give you any physical intuition for the imaginary solutions, because they are just that, imaginary. They're just a computational tool. All I can say is that an imaginary solution $e^{i\omega t}$ does correspond to a real solution oscillating with angular frequency ω. But we do need to understand the details of what I mean by "corresponds".

Example: Let's look back at your homework problem:



N=6 beads on springs

You could have written down a 6×6 K matrix, and diagonalized it to find the motion. This is a horrible algebraic mess. So instead you wrote down the symmetry matrix S, and without too much work, found the N=6 eigenvalues:

$$\beta_n = e^{i\frac{2\pi}{N}\cdot n} \qquad \text{for } n=1\ldots 6$$

and the corresponding N=6 eigenvectors:

$$b^{(n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot n} \\ e^{i\frac{2\pi}{N}\cdot 2n} \\ e^{i\frac{2\pi}{N}\cdot 3n} \\ \vdots \\ e^{i\frac{2\pi}{N}\cdot Nn} \end{pmatrix} \qquad \text{for } n=1\ldots 6$$

We know (because K and S commute, i.e. KS=SK) that these N=6 $b^{(n)}$'s must also be eigenvectors of our K matrix.

But the problem asked for the <u>real</u> eigenvectors of the K matrix. And these 6 $b^{(n)}$'s are patently imaginary.

So what do we do?

$\boxed{\text{Practical answer}}$ Take the real and imaginary parts of our $b^{(n)}$'s.

This is all you really need to remember.

But to alleviate some more confusion, lets look in more detail to see <u>why</u> this practical strategy works

<u>Fact</u>: $b^{(n)}$'s are eigenvectors of real K matrix with real eigenvalues $-\omega_n^2$

Let's prove 2 simple and relevant theorems:

<u>Thm</u>: If v is an eigenvector of a real matrix M with a real eigenvalue c, then v^* (complex conjugate of v) is also an eigenvector of M with the same eigenvalue c.

<u>Proof</u>: $M v^* = (Mv)^* = (cv)^* = c v^*$

can pull M inside the complex conjugate operator because M is real, so $M^* = M$

can pull c outside the complex conjugate operator because c is real, so $c^* = c$

$$Mv^* = cv^* \Rightarrow v^* \text{ is an eigenvector of } M \text{ with eigenvalue } c$$

Consequences of this theorem to our mass-spring problem:

If $b^{(n)}$ is an eigenvector of K, then $b^{(n)*}$ must also be an eigenvector of K.

But we already know all $N=6$ eigenvectors of K.

So for each $b^{(n)}$, $b^{(n)*}$ must also be an eigenvector. Some of the $b^{(n)}$'s must be complex conjugates of each other!

Note that for $n = N/2 = 3$ and $n = N = 6$, the $b^{(n)}$'s are already real, so they are trivially their own complex conjugates.

$$b^{(3)} = b^{(3)*} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad b^{(6)} = b^{(6)*} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For other values of n, we have

$$b^{(n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot n} \\ e^{i\frac{2\pi}{N}\cdot 2n} \\ \vdots \\ e^{i\frac{2\pi}{N}\cdot Nn} \end{pmatrix}$$

$$b^{(N-n)} = \begin{pmatrix} e^{i\frac{2\pi}{N}\cdot(N-n)} \\ e^{i\frac{2\pi}{N}(2N-2n)} \\ \vdots \\ e^{i\frac{2\pi}{N}(N^2-Nn)} \end{pmatrix} = \begin{pmatrix} e^{i\frac{2\pi}{N}N} e^{i\frac{2\pi}{N}(-n)} \\ e^{i\frac{2\pi}{N}2N} e^{i\frac{2\pi}{N}(-2n)} \\ \vdots \\ e^{i\frac{2\pi}{N}N^2} e^{i\frac{2\pi}{N}(-Nn)} \end{pmatrix} = \begin{pmatrix} e^{2\pi i} e^{-i\frac{2\pi}{N}n} \\ e^{4\pi i} e^{-i\frac{2\pi}{N}\cdot 2n} \\ \vdots \\ e^{N\pi i} e^{-i\frac{2\pi}{N}Nn} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{2\pi}{N}n} \\ e^{-i\frac{2\pi}{N}2n} \\ \vdots \\ e^{-i\frac{2\pi}{N}Nn} \end{pmatrix} = b^{(n)*}$$

So in general, $b^{(n)} = b^{(N-n)*}$.

The $b^{(n)}$'s have complex eigenvalues of matrix S, so the above theorem does not apply. But the $b^{(n)}$'s have real eigenvalues of matrix K (because these are the real frequencies of the system), so the above theorem does apply.

So $b^{(n)} = b^{(N-n)*}$ and these 2 eigenvectors share the same eigenvalue of K, with value $-\omega_n^2$.

Theorem. If 2 eigenvectors v_1 and v_2 of matrix M share an eigenvalue c, then any linear combination of v_1 and v_2 is also an eigenvector of M with the same eigenvalue c. (Note: this second theorem does not rely on M and c being real.)

Proof: $M(av_1 + bv_2) = aMv_1 + bMv_2 = acv_1 + bcv_2 = c(av_1 + bv_2)$
$\Rightarrow av_1 + bv_2$ is an eigenvector of M with eigenvalue c.

(Note is this true for any general eigenvectors v_1 and v_2 of M? No, v_1 and v_2 must share the eigenvalue c.)

Consequences of this theorem to our mass-spring problem:

We already know from the 1st theorem that $b^{(n)}$ and $b^{(n)*}$ are both eigenvectors of the real K, sharing the same real eigenvalue $-\omega_n^2$.

$\Rightarrow$ any linear combo of $b^{(n)}$ and $b^{(n)*}$ is also an eigenvector of K

$\Rightarrow$ in particular, $\dfrac{b^{(n)} + b^{(n)*}}{2}$ and $\dfrac{b^{(n)} - b^{(n)*}}{2i}$ are eigenvectors of K

$\Rightarrow$ We get back to our original practical answer: take the Re and Im parts of each $b^{(n)}$.

Mathematically, what facts did we have to rely on to get to this point? Only: K is real and $-\omega_n^2$ is real

$\Rightarrow$ Quite generally, for any real world problem in which you come up with some set of imaginary eigenvectors, you can just take the Re and Im parts.

- How do we actually know, mathematically, that the normal modes are eigenvectors of the K matrix? Is that just how they are defined? Why does this make sense physically?

What a normal mode means physically is: a mode in which all masses oscillate with the same amplitude. So we want a vector of amplitudes such that no matter which force equation we put it into, we get the same frequency ω.

$$F = ma$$

$$k_1^{(i)} x_1 + k_2^{(i)} x_2 + \ldots + k_N^{(i)} x_N = m \frac{d^2 x_i}{dt^2}$$

$$k_1^{(j)} x_1 + k_2^{(j)} x_2 + \ldots + k_N^{(j)} x_N = m \frac{d^2 x_j}{dt^2}$$

We want an eigenvector $\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ of K such that no matter which force equation we plug in to (which row of the K matrix) we get the same factor of $-\omega^2$ from the derivative on the right (i.e. all masses oscillate with the same frequency).

- How is the resonance frequency of an entire system related to the resonance frequency of the individual components (i.e. pendulum's resonance frequency)?

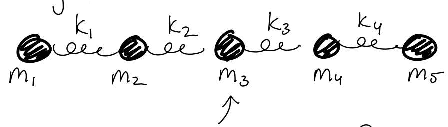
The whole system has normal modes at certain frequencies, but these aren't really a "resonance frequency" of the whole system – i.e. left to its own devices there is no special tendency of the system to want to oscillate in most of its normal modes.

Usually there is one normal mode frequency which equals the resonance frequency of the individual components (i.e. all individual components move together in synch) but all other normal mode frequencies are different.

- In a typical masses & springs problem, are the masses only affected by their neighbors? Or do we have to consider all of them?

If you want to write down all the forces on a mass, there are really only 3 types you have to worry about: electromagnetic, mechanical, and gravity (until you get to sub-atomic particles and include strong & weak forces)

Electromagnetic & gravity can act from a distance, but mechanical can act only if it's touching. So, really, a mass is only affected by the springs that are touching it (i.e. the mass is not even affected by other masses). You do have to take into account the full length of all touching springs, i.e. the positions of the nearest neighbor masses do matter.



If you want to write down the forces on this guy, then only the lengths of springs k_2 & k_3 matter. So the positions of m_2 & m_4 are relevant, because they determine the lengths of the relevant springs. But the positions of m_1 & m_5 are irrelevant.

- Why exactly are the coefficients in the Fourier series the "overlap" or dot product – not why this works for the coefficients but how you can actually take the dot product with an infinite sum.

The dot product distributes, just like regular multiplication.

$$\vec{v}_1 \cdot (a\vec{v}_2 + b\vec{v}_3) = a\vec{v}_1 \cdot \vec{v}_2 + b\,\vec{v}_1 \cdot \vec{v}_3$$

$$f(t) \circ g(t) \equiv \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt$$

$$f(t) \circ \sum_{i=1}^{\infty} g_i(t) = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} f(t) g_i^*(t)\, dt$$

- How can a wave distort with part of it moving faster than the speed of light?

Information

What is information? lots of technical definitions ...

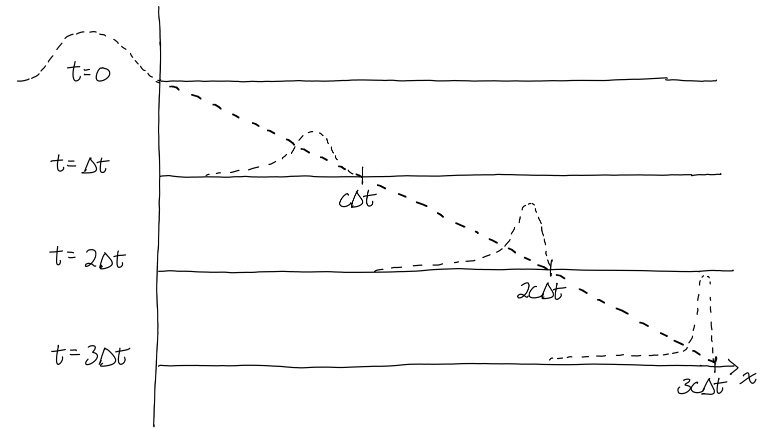
For our purposes, we'll just think of it as a notification that something happened. If an event at point A sends a signal (a wave pulse) to point B, a distance d away, an observer at B can't possibly know that anything has happened at A until at least the very leading edge of this pulse has arrived. The leading edge cannot travel faster than the speed of light, c.

Observations:

1. Of course there may be additional, more complicated information contained in the rest of the pulse, such as peak height, pulse shape, carrier frequency, etc. But since the leading edge can travel no faster than c, of course this additional info can travel no faster than its own leading edge, therefore no faster than c.
2. If the notification of the event at A could travel faster than c, then it would be possible to find a reference frame in which causality is violated. For example, suppose the notification of event #1 at A travels to point B at a speed $\frac{3}{2}c$ (measured in the fixed reference frame of A and B). Suppose the arrival of the notification at B immediately triggers event #2 at B. Now if you go into a reference frame moving along the line from A to B at sufficient speed, say 0.9c, event #2 will occur before the event #1 which caused it.

"Nothing travels faster than the speed of light, with the possible exception of bad news, which obeys its own set of laws." -Douglas Adams

3. It is possible to have $v_{phase} > c$
If a wave is at a single fixed frequency, it always was and always will be, so it carries no new information from point A to point B. If there is any change to the wave, this change will carry information, but this change will contain other frequency components, and the leading edge of this change can travel no faster than c.
4. It is also possible for the peak of a pulse to move faster than c (i.e. it is possible to have $v_{group} > c$)



But note that the leading edge cannot move faster than c.

Question: How do we analyze/prove these statements about the travel of leading edges, peak heights, etc?

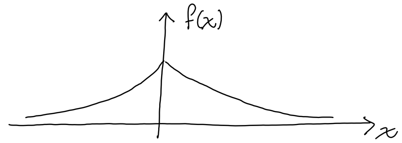
Answer: Proofs are long & algebraically gross, but all hinge on Fourier transforms.

$$\boxed{\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{-i\omega t}\, d\omega \\ \tilde{F}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \end{aligned}}$$

← inverse Fourier transform

← Fourier transform

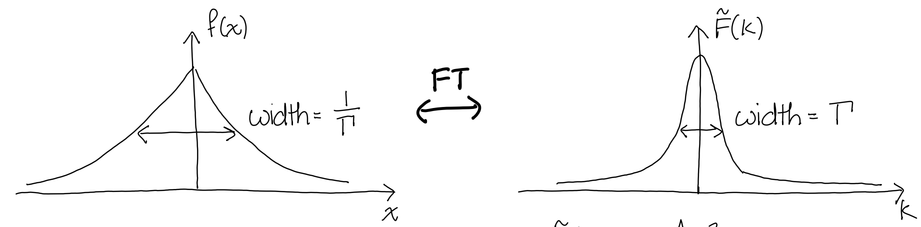
Example: $f(x) = Ae^{-\Gamma|x|/2}$



Take the Fourier transform to find $\tilde{F}(k)$:

$$\begin{aligned}
\tilde{F}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Ae^{-\Gamma|x|/2} e^{ikx}\, dx \\
&= \frac{A}{2\pi}\left\{ \int_{-\infty}^{0} e^{\left(\frac{\Gamma}{2}+ik\right)x}\, dx + \int_{0}^{\infty} e^{\left(-\frac{\Gamma}{2}+ik\right)x}\, dx \right\} \\
&= \frac{A}{2\pi}\left\{ \frac{1}{\frac{\Gamma}{2}+ik} e^{\left(\frac{\Gamma}{2}+ik\right)x}\Big|_{-\infty}^{0} + \frac{1}{-\frac{\Gamma}{2}+ik} e^{\left(-\frac{\Gamma}{2}+ik\right)x}\Big|_{0}^{\infty} \right\} \\
&= \frac{A}{2\pi}\left\{ \frac{1}{\Gamma/2+ik} - \frac{1}{-\Gamma/2+ik} \right\} \\
&= \frac{A}{2\pi}\left\{ \frac{(\Gamma/2-ik)+(\Gamma/2+ik)}{(\Gamma/2)^2+k^2} \right\} \\
&= \frac{A}{\pi}\left[\frac{\Gamma/2}{(\Gamma/2)^2+k^2} \right]
\end{aligned}$$

The Fourier transform of an exponential decay is a Lorentzian



$$\tilde{F}(k=0) = \frac{A}{\pi}\frac{2}{\Gamma}$$

$$\tilde{F}\left(k=\frac{\Gamma}{2}\right) = \frac{A}{\pi}\frac{1}{\Gamma} = \frac{1}{2}\tilde{F}(k=0)$$

A wider peak in real space leads to a narrower peak in frequency space, and vice versa.

Note: we could also compute the inverse Fourier transform to get back where we started:

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \tilde{F}(k) e^{-ikx}\, dk \\
&= \int_{-\infty}^{\infty} \frac{A}{\pi}\left[\frac{\Gamma/2}{(\Gamma/2)^2+k^2}\right] e^{-ikx}\, dk
\end{aligned}$$

Integral is somewhat uglier → ask Mathematica to do it!

Find that:

$$\int_{-\infty}^{\infty} \frac{A}{\pi}\left[\frac{\Gamma/2}{(\Gamma/2)^2+k^2}\right] e^{-ikx}\, dk = Ae^{-\Gamma|x|/2}$$

exactly as we expected!

Extreme example: **δ-Function**

$f(x) = \delta(x)$

$\uparrow$

Special function defined such that

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \quad \text{but } \delta(x) = \begin{cases} \infty & \text{at } x=0 \\ 0, & \text{everywhere else} \end{cases}$$

$$\tilde{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{ikx}dx = \frac{1}{2\pi}\left(e^{ikx}\right)\Big|_{x=0} = \frac{1}{2\pi}$$

A function that is infinitely sharp in real space requires an infinite spread of frequencies to represent it.

Inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \underbrace{e^{-ikx}}\,dk$$

For every value of x except $x=0$, this is oscillatory and thus completely cancels itself out when integrating over all k.

$\Rightarrow f(x) = 0$ for $x \neq 0$

For $x=0$, we have $f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi}dk \rightarrow \infty$

So indeed, we have $f(x) = \delta(x)$ from the inverse Fourier transform, as expected.