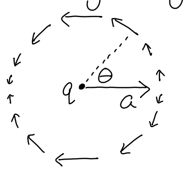
Physics 15c (Hoffman)
Lecture # 19
Tues, Nov 16, 2010

Interference & Diffraction

Last time:

* Accelerating charges



$$E_{radiated} = \frac{\mu_0 q}{4\pi}\frac{\sin\theta}{r}\, a\left(t-\frac{r}{c}\right)$$

$$\text{Poynting: } S = \frac{\mu_0 q^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\left[a\left(t-\frac{r}{c}\right)\right]^2$$

* Larmor formula: integrate S over spherical area
$$P = \frac{\mu_0 q^2}{6\pi c}\left[a\left(t-\frac{r}{c}\right)\right]^2$$
* Rayleigh scattering: calculate Larmor formula for the simple harmonic acceleration of an oscillating charge excited by EM waves: $P = \frac{\mu_0 q^2 \omega^4 x_0^2}{12\pi c}$

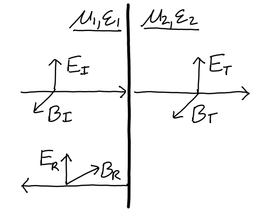
  → point is: scattering goes like ω^4
* Light in matter - derivation of index of refraction n
* Mirages - when n changes with hot air

Goals for today:

* Thin film interference
* Newton's rings
* Double-slit interference
* Many-slit interference → diffraction grating, crystallography
* Wide slit diffraction
* Connection to Fourier transforms

Interference

We studied reflection at a boundary between insulators as a function of incident angle.



Let's back up to normal incidence
⇒ continuity of $\vec{E}$ and $\vec{H}$ give:

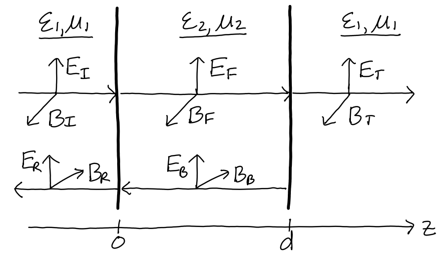
$$E_R = \frac{n_1-n_2}{n_1+n_2}E_I$$

$$E_T = \frac{2n_1}{n_1+n_2}E_I$$

$$R = \left(\frac{n_1-n_2}{n_1+n_2}\right)^2 ;\quad T = \frac{4n_1n_2}{(n_1+n_2)^2}$$

(where we ditched Z in favor of the more commonly used n, assuming $\mu_2 \approx \mu_1 \approx \mu$)

Let's consider 2 (or more) boundaries in series
e.g. consider a window pane



At z=0,

① $E_I + E_R = E_F + E_B$ (continuity of E$^{\parallel}$)

② $\frac{E_I - E_R}{Z_1} = \frac{E_F - E_B}{Z_2}$ (continuity of H$^{\parallel}$)

At z=d,

③ $E_F e^{ik_2d} + E_B e^{-ik_2d} = E_T$

④ $\frac{E_F e^{ik_2d} - E_B e^{-ik_2d}}{Z_2} = \frac{E_T}{Z_1}$

Exponents appear because:
$E_F(z,t) = E_F e^{-i(\omega t - k_2 z)}$
$E_B(z,t) = E_B e^{-i(\omega t - k_2 z)}$

Solve 4 eqns for 4 unknowns E_T, E_R, E_F, E_B

$$E_R = \left(\frac{(z_1^2 - z_2^2)(e^{ik_2d} - e^{-ik_2d})}{(z_1+z_2)^2 e^{-ik_2d} - (z_1-z_2)^2 e^{ik_2d}}\right) E_I$$

reflectivity is $|\text{this}|^2$

$$E_T = \left(\frac{4z_1z_2e^{-ik_2d}}{(z_1+z_2)^2 e^{-ik_2d} - (z_1-z_2)^2 e^{ik_2d}}\right) E_T$$

transmittivity is $|\text{this}|^2$

$$E_F = \left(\frac{2z_2(z_1+z_2)e^{-ik_2d}}{(z_1+z_2)^2 e^{-ik_2d} - (z_1-z_2)^2 e^{ik_2d}}\right) E_T$$

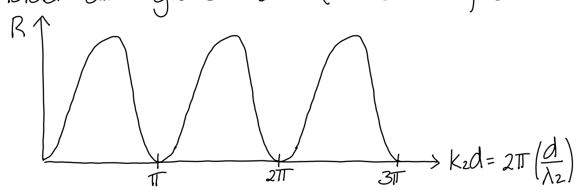
$$E_B = \left(\frac{2z_2(z_1-z_2)e^{ik_2d}}{(z_1+z_2)^2 e^{-ik_2d} - (z_1-z_2)^2 e^{ik_2d}}\right) E_T$$

$$R = \left|\frac{E_R}{E_I}\right|^2 = \frac{2(z_1^2-z_2^2)^2 \sin^2(k_2d)}{8z_1^2z_2^2 + 2(z_1^2-z_2^2)^2\sin^2(k_2d)}$$

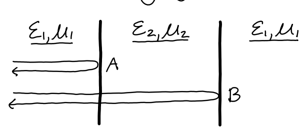
⇒ reflectivity oscillates as a function of k_2d

⇒ reflectivity vanishes at $k_2d = m\pi$

Consider air → glass → air ($z_1 = 377\,\Omega$; $z_2 = 250\,\Omega$)



Whenever $2d$ is an integer # of wavelengths λ_2 in medium 2 the reflectivity goes to zero.



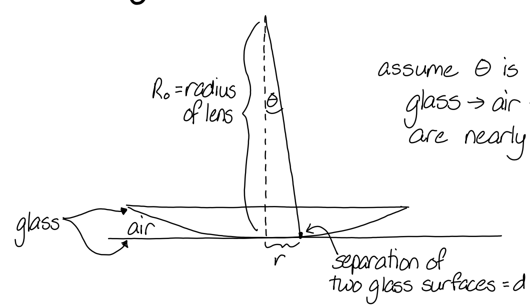
Compare reflection paths A and B: path difference is $2d$.

That's weird! When path difference is an integer # of wavelengths, we get destructive interference ?!?!

Right, because remember we have one hard → soft boundary and one soft → hard boundary, so one of them must flip the sign of the wave (phase shift it by π).

⇒ path difference of $m\lambda$ gives destructive interference

Newton's Rings

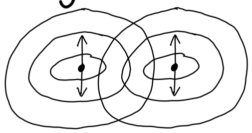


assume θ is so small that glass → air → glass surfaces are nearly parallel

→ can compute radii r at which reflection vanishes

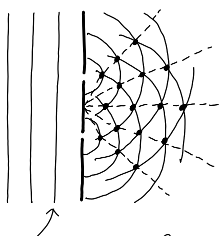
General note on interference: Always need to look at the amplitudes of all the relevant waves, including phase information. Add these all up. Only after adding everything do you square to get the intensity.

Two-Body Interference



e.g. 2 charged particles each radiate EM waves
$\Rightarrow$ what's the total?

Huygen's principle tells us that a plane wave passing through a double slit gives us a very similar situation.



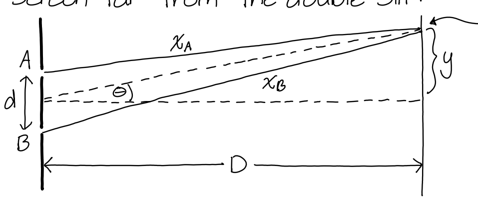
Waves going through the holes spread spherically.

Young's double slit experiment (1801-1803) demonstrated the wave nature of light

Huygen: wave fronts are $\perp$ to direction of propagation; draw wave crests a distance λ apart

$\Rightarrow$ Wherever crests intersect = constructive interference.
In between, we have a crest from one wave adding to a trough from the other = destructive interference
$\Rightarrow$ Wave amplitude varies depending on direction.

Put a screen far from the double slit:



E-field here comes from E_A and E_B

$|E_A| \propto \frac{1}{x_A}$ and $|E_B| \propto \frac{1}{x_B}$

but if θ is small and $x_A \approx x_B \gg d$
then we ignore the difference in the magnitudes of $|E_A|$ and $|E_B|$

But the difference in phase matters more

$$E_A = E_0 e^{i(kx_A - \omega t)} \quad ; \quad E_B = E_0 e^{i(kx_B - \omega t)}$$

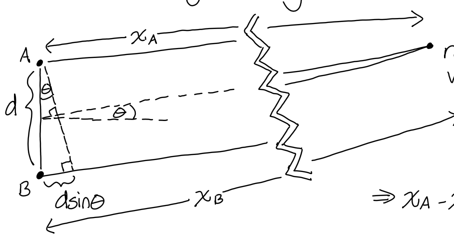
$\Rightarrow$ phase differs by $k(x_A - x_B)$

$$E = E_A + E_B = E_0\left(e^{ikx_A} + e^{ikx_B}\right)e^{-i\omega t}$$

$$= \underbrace{2E_0 \cos\left(\frac{k(x_A - x_B)}{2}\right)}_{\text{amplitude}} e^{i\left(\frac{k(x_A+x_B)}{2} - \omega t\right)}$$

$$\Rightarrow S \propto \left(E_0 \cos\left(\frac{k(x_A - x_B)}{2}\right)\right)^2$$

Compute $x_A - x_B$ from geometry



rays converge very far away

$\Rightarrow x_A - x_B = d\sin\theta$

$$I = I_0 \cos^2\left(\frac{k(x_A - x_B)}{2}\right) = I_0 \cos^2\left(\frac{kd\sin\theta}{2}\right) = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

(If we take into account the $1/r$ dependence of amplitude, then the exact solution is much uglier. But as long as $d\sin\theta \ll x_A, x_B$ then we have a very good approximation.
If $x_A, x_B > 5d$, then it's pretty much exact.)

Consider the pattern on the screen as a function of y:

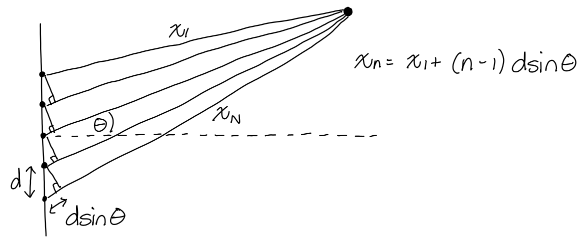
$$\sin\theta \approx \tan\theta = \frac{y}{D}$$

$$\Rightarrow I = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \approx I_0 \cos^2\left(\frac{\pi d y}{\lambda D}\right)$$

$\Rightarrow$ simple $\cos^2$ pattern in y (for small θ, large D)

N slits

(Sneak preview: we are on our way to taking $N \to \infty$ and $d \to 0$, i.e. infinite number of infinitesimally close slits
$\Rightarrow$ this will allow us to understand a wide slit, aka a lens)



Add up $\vec{E}$ from all sources:

$$E = \sum_{n=1}^{N} E_0\, e^{i(kx_n - \omega t)} = E_0\, e^{i(kx_1-\omega t)} \sum_{n=1}^{N} e^{ik(n-1)d\sin\theta}$$

$$= E_0\, e^{i(kx_1-\omega t)} \sum_{n=1}^{N} \left(e^{ikd\sin\theta}\right)^{n-1}$$

There's a polynomial identity we can use:

$$\sum_{n=1}^{N} x^{n-1} = \frac{x^N - 1}{x-1} \quad \leftarrow \text{can prove this by induction}$$

(Quick check for $N=2$: $1 + x = \frac{x^2-1}{x-1}$ ✓)

$$E = E_0\, e^{i(kx_1-\omega t)} \sum_{n=1}^{N}\left(e^{ikd\sin\theta}\right)^{n-1} = E_0\, e^{i(kx_1-\omega t)}\, \frac{e^{ikNd\sin\theta}-1}{e^{ikd\sin\theta}-1}$$

$$= E_0\, e^{i(kx_1-\omega t)} \left\{ \frac{e^{ik\frac{N}{2}d\sin\theta}\left(e^{ik\frac{N}{2}d\sin\theta} - e^{-ik\frac{N}{2}d\sin\theta}\right)}{e^{ik\frac{1}{2}d\sin\theta}\left(e^{ik\frac{1}{2}d\sin\theta} - e^{-ik\frac{1}{2}d\sin\theta}\right)} \right\}$$

$$= \underbrace{E_0\, e^{i(kx_1-\omega t)}\, e^{ik\left(\frac{N-1}{2}\right)d\sin\theta}}_{\text{ultimately we're interested in } I\text{, so phase doesn't matter}}\ \frac{\sin\left(\frac{kNd\sin\theta}{2}\right)}{\sin\left(\frac{kd\sin\theta}{2}\right)}$$

$$|E| = |E_0|\, \frac{\sin(Nx)}{\sin x} \quad \text{where } x \equiv \frac{kd\sin\theta}{2}$$

What does this look like?

$\rightarrow$ it's periodic in x (numerator repeats every $\Delta x = \frac{2\pi}{N}$

but denominator repeats less frequently, only every $\Delta x = 2\pi$, so overall period is 2π)

$\rightarrow$ but what if $x = 0$?

$$\lim_{x\to 0}\left(\frac{\sin Nx}{\sin x}\right) = \lim_{x \to 0}\left(\frac{Nx + \ldots}{x + \ldots}\right) = N$$

Taylor expand numerator & denominator for small N

For a given N, compute I relative to $\theta = 0$ (which corresponds to $x=0$)

$$\frac{I(\theta)}{I(0)} = \left(\frac{\frac{\sin(Nx)}{\sin x}}{N}\right)^2 = \left(\frac{\sin(Nx)}{N\sin x}\right)^2$$

$\rightarrow$ this has big peaks (where denominator $\to 0$) at $x = 0, \pi, 2\pi, \ldots = \frac{kd\sin\theta}{2} = \frac{\pi d\sin\theta}{\lambda}$

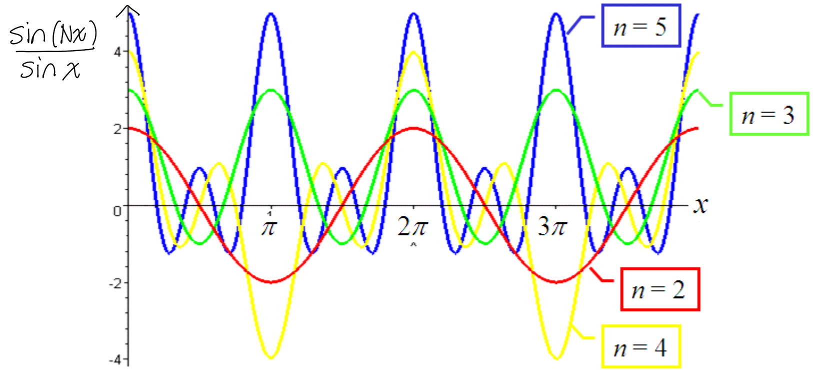
$\Rightarrow \sin\theta = 0, \frac{\lambda}{d}, \frac{2\lambda}{d}, \ldots$ etc.

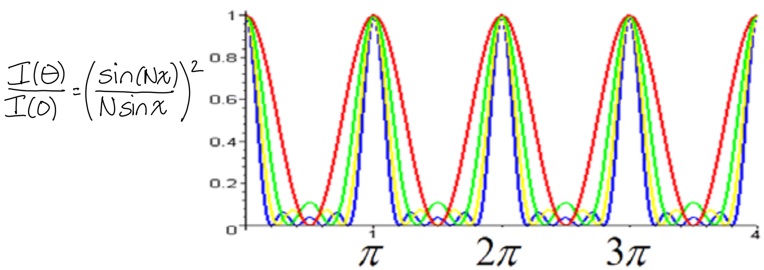
$\rightarrow$ peaks get narrower as N increases

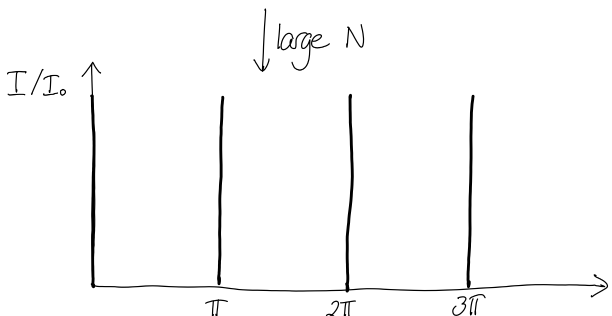
(this is because the numerator repeats every $\Delta x = \frac{2\pi}{N}$

so the numerator goes $\to 0$ faster when $N\uparrow$ and $\Delta x \downarrow$)

If $N \to \infty$, the width vanishes $\Rightarrow$ we get δ-function peaks







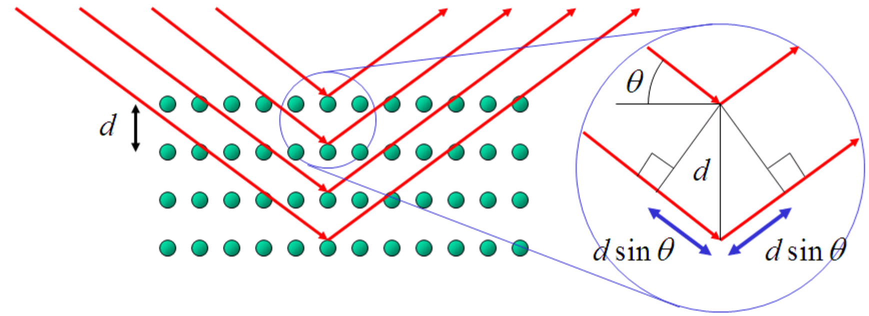
Diffraction grating

= a plate with fine strips, like a transparent film with regular scratches, so light only gets through between scratches

Light is "diffracted" at angle θ only if $\sin\theta = \frac{n\lambda}{d}$

Crystallography

A crystal is a regular array of molecules, each of which can scatter light

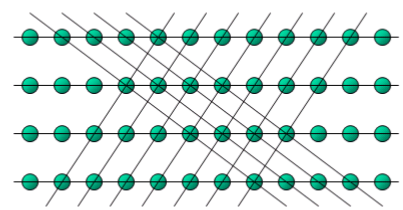


Constructive interference occurs between different scattering sites if the path difference is an integer multiple of λ.

This leads to the so-called "Bragg condition": $2d\sin\theta = n\lambda$

So if we set up an experiment to scatter light at a fixed λ off a crystal, and we can vary the angle and look for peaks in the intensity of scattered light, then we can use the λ and θ information to figure out the lattice spacing d of the crystal.

Actually, there are many different planes within a crystal:



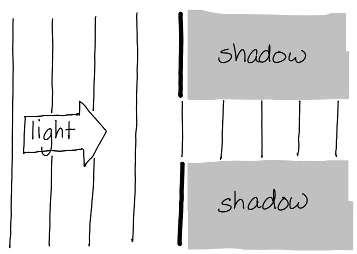
So by varying the angle, we can reconstruct entire crystal structure.

Note: need λ comparable to d to make this work $\Rightarrow$ use x-rays or electrons

So far, we've considered only infinitesimal slits (point sources)

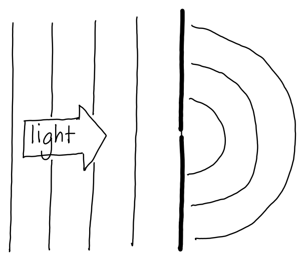
Now we consider slits of finite width:

Wide slit



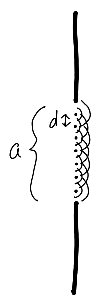
observer here does not see light

Narrow slit



observer here sees light!

This is weird!
→ try to understand it via Huygen's principle
→ consider every point in the wide slit to be a point source



N = # of points → ∞
d = distance between pts → 0
→ but need to keep $Nd = a$ = width of slit

We know the answer for any N:

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin(N\chi)}{N\sin\chi}\right)^2 \quad \text{where } \chi \equiv \frac{kd\sin\theta}{2}$$

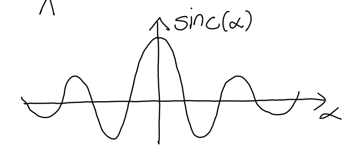
If $d = a/N$, then we write:

$$\chi = \frac{ka\sin\theta}{2N} \Rightarrow N\chi = \frac{ka\sin\theta}{2} \Rightarrow \frac{\sin(N\chi)}{N\sin\chi} = \frac{\sin\left(\frac{ka\sin\theta}{2}\right)}{N\sin\left(\frac{ka\sin\theta}{2N}\right)}$$

$$\lim_{N\to\infty}\left(\frac{\sin\left(\frac{ka\sin\theta}{2}\right)}{N\sin\left(\frac{ka\sin\theta}{2N}\right)}\right) = \lim_{N\to\infty}\left(\frac{\sin\left(\frac{ka\sin\theta}{2}\right)}{N\left(\frac{ka\sin\theta}{2N}\right)}\right) = \frac{\sin\left(\frac{ka\sin\theta}{2}\right)}{\frac{ka\sin\theta}{2}}$$

$$\Rightarrow \frac{I(\theta)}{I(0)} = \left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \text{where } \alpha \equiv \frac{ka\sin\theta}{2} = \frac{\pi a\sin\theta}{\lambda}$$

$\frac{\sin\alpha}{\alpha}$ is called a "sinc" function $= \mathrm{sinc}(\alpha)$

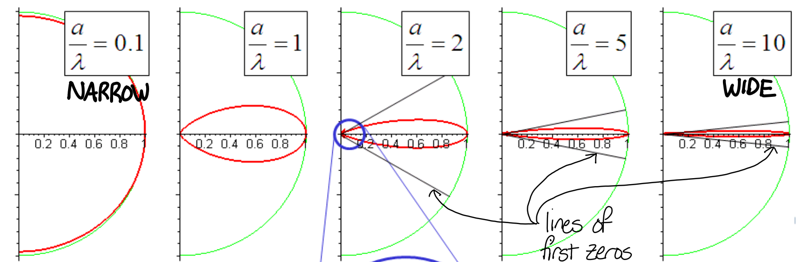


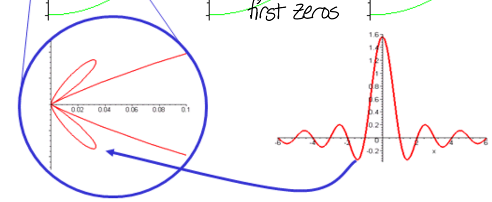
Zeros occur where $\alpha = \ldots -3\pi, -2\pi, -\pi, \pi, 2\pi, 3\pi, \ldots$
(note: when $\alpha = 0$, $\mathrm{sinc}(\alpha)$ is maximum)

⇒ zeros where $\alpha = \frac{\pi a\sin\theta}{\lambda} = n\pi$ (for all integers n, except $n=0$)

⇒ zeros occur for values of θ where $\sin\theta = n\lambda/a$ (except $n=0$)

Plot $\frac{I(\theta)}{I(0)}$ for different slit thicknesses:





Quiz: Where is the line of first zeros for $a = \lambda$?