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Physics 15c (Hoffman)
Lecture #17
Tues, November 2, 2010

Electromagnetic Radiation (from accelerating charges)

Last time:

* Conductors: skin depth, $1/d = \mathcal{K} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)^{1/2}$
* Insulators: replace $\varepsilon_0 \to \varepsilon, \mu_0 \to \mu \Rightarrow c_W = \frac{c}{n}$ where $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$
* Huygen's principle: draw successive wavefronts using point sources on previous wavefront
* Snell's law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$
* Total internal reflection: $\theta_c = \sin^{-1}(n_1/n_2)$ [where $n_2 > n_1$]
* Reflection & refraction @ non-normal incidence → Fresnel coeffs
  Vertical polarization ($\vec{B}$ parallel to surface) $E_R = \frac{\alpha-\beta}{\alpha+\beta}$ $E_T = \frac{2}{\alpha+\beta}$
  [where $\alpha = \cos\theta_2/\cos\theta_1$ and $\beta = Z_1/Z_2$]
* Brewster's angle: angle at which vertically polarized light is not reflected
  $\tan\theta_B = \frac{n_2}{n_1}$; At Brewster's angle, $\theta_1 + \theta_2 = 90°$
  i.e. this is Brewster's angle θ_B

Goals for today:

* EM radiation from accelerating charge → explain Brewster's angle!
* power from oscillating charge
* Rayleigh scattering - why is the sky blue? why is a sunset red?

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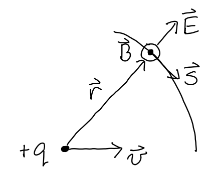
Radiation of Accelerating Charge

This derivation comes from the appendix B to Purcell E&M

Consider moving a point charge $+q$ with constant velocity $\vec{v}$
→ what is the field at an arbitrary location $\vec{r}$ away from $+q$?

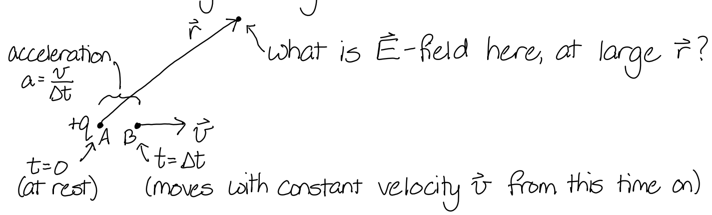
If the charge is moving at constant $\vec{v}$ (and if it has been moving at constant $\vec{v}$ for time T long enough so that $cT > |\vec{r}|$, i.e. so that the point at $\vec{r}$ "knows" by special relativity that the charge is moving at constant $\vec{v}$)
Then the $\vec{E}$-field at $\vec{r}$ points along $\vec{r}$; $\vec{B}$ is perpendicular as if $+q$ were current in a wire along $\vec{v}$.



Poynting vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$
⇒ no power is radiated outward

BUT, if charge accelerates, this changes...

Now consider a charge initially at rest

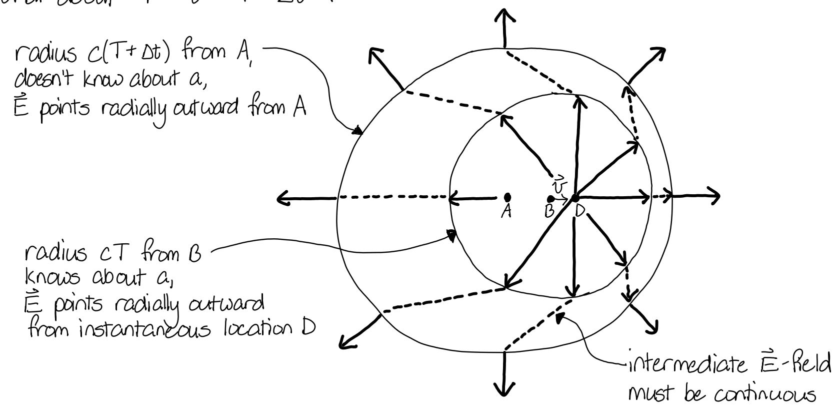


We want to know $\vec{E}$ at large distance $\vec{r}$.
let $T = \frac{|\vec{r}|}{c}$ and assume $T \gg \Delta t$
(i.e. assume that $\vec{r}$ is far enough that info takes longer to get to $\vec{r}$ than the short duration of the acceleration)

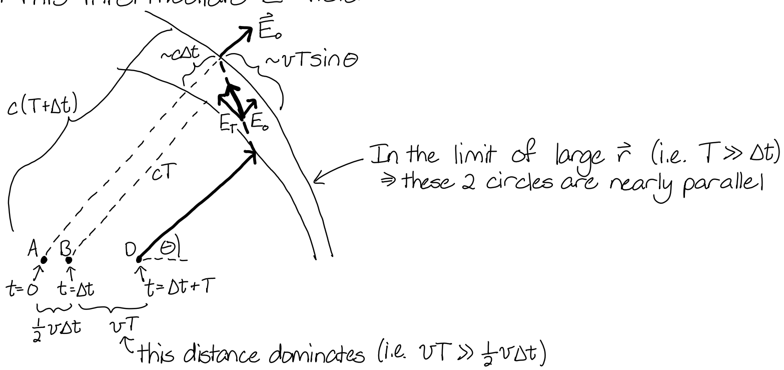
Before $t = T$: $\vec{r}$ doesn't know about acceleration
⇒ $\vec{E}$ points radially outward from original location A

After $t = T + \Delta t$: $\vec{r}$ knows about constant $\vec{v}$
⇒ $\vec{E}$ points radially outward from new instantaneous location

What about $T < t < T+\Delta t$?



Zoom in on this intermediate $\vec{E}$-field:



Use some trig: $\frac{E_T}{E_0} \sim \frac{vT\sin\theta}{c\Delta t}$ (in the limit $v \ll c$ and $\Delta t \ll T$)

$$\Rightarrow E_T = E_0 \frac{vT\sin\theta}{c\Delta t} = \frac{q}{4\pi\varepsilon_0 r^2}\frac{vT\sin\theta}{c\Delta t}$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\frac{vr\sin\theta}{c^2\Delta t}$$ (remember $T = \frac{r}{c}$ = time for $\vec{r}$ to know about a)

$$= \frac{q\sin\theta}{4\pi\varepsilon_0 r c^2}\frac{v}{\Delta t} = \frac{qa\sin\theta}{4\pi\varepsilon_0 r c^2}$$ (because $v/\Delta t = a$)

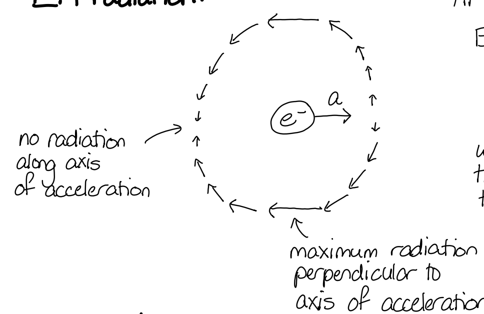
⇒ E_T is proportional to acceleration a, and has angular dependence $\sin\theta$.

Compare: $E_T = \frac{qa\sin\theta}{4\pi\varepsilon_0 c^2 r}$ ← falls off like $\frac{1}{r}$ $\qquad E_0 = \frac{q}{4\pi\varepsilon_0 r^2}$ ← falls off like $\frac{1}{r^2}$

⇒ at large $\vec{r}$, $E_T \gg E_0$, so radiated $\vec{E}$-field is transverse

Why Brewster's Angle?

EM radiation:

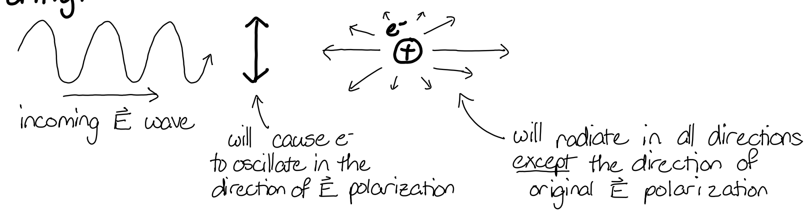


At large $\vec{r}$:

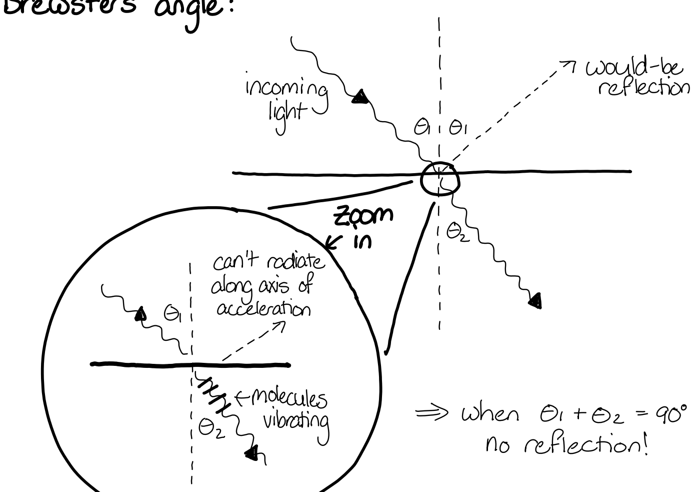
$$E \approx E_T = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\sin\theta}{r}a = \frac{\mu_0 q}{4\pi}\frac{\sin\theta}{r}\underbrace{a\left(t-\frac{r}{c}\right)}$$

write as a function of time, to emphasize that the point $\vec{r}$ only knows about the acceleration that took place a time $T = r/c$ ago

Scattering:



Brewster's angle:



⇒ when $\theta_1 + \theta_2 = 90°$
no reflection!

Radiated Power

Poynting vector: $\vec{S} = \frac{1}{\mu_0} E_T B_T$

$$E_T = \frac{\mu_0 q}{4\pi} \frac{\sin\theta}{r} a\left(t - \frac{r}{c}\right)$$

$$B_T = \frac{E_T}{c} = \frac{\mu_0 q}{4\pi c} \frac{\sin\theta}{r} a\left(t - \frac{r}{c}\right)$$

$$\Rightarrow S = \frac{\mu_0 q^2}{16\pi^2 c^2} \frac{\sin^2\theta}{r^2} \left[a\left(t - \frac{r}{c}\right)\right]^2$$

↑ good! power falls off like $1/r^2$, as expected!

Larmor Formula

What's the total power radiated by an accelerating charge?

→ integrate S over the surface of a sphere at radius $|\vec{r}|$

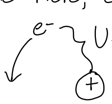
$$P = \int_0^{2\pi} d\phi \int_0^{\pi} S r^2 \sin\theta \, d\theta$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} = \frac{\mu_0 q^2 a^2}{6\pi c}$$

⇒ Accelerating charge loses energy at a rate proportional to acceleration squared!

Bohr atom

As a side note, consider the classical idea of an e^- "orbiting" an atom:



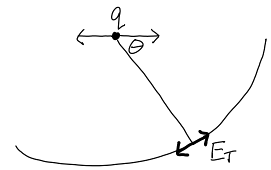
$U = -\frac{q^2}{4\pi\varepsilon_0 r}$ = potential energy

⇒ energy would decay in nanoseconds and e^- would crash into ⊕

⇒ Bohr says e^- must be waves in quantized orbits

Oscillating Charge

$x = x_0 e^{-i\omega t} \Rightarrow a = \frac{d^2x}{dt^2} = -\omega^2 x_0 e^{-i\omega t}$



$$E_T = \frac{\mu_0 q}{4\pi} \frac{\sin\theta}{r} a\left(t - \frac{r}{c}\right) = -\frac{\mu_0 q \omega^2 x_0}{4\pi} \frac{\sin\theta}{r} e^{-i\omega\left(t - \frac{r}{c}\right)}$$

↑ Check sign:
$\vec{a}$ points opposite to $\vec{x}$ (restoring force)
but $\vec{E}_T$ points opposite to $\vec{a}$

So it makes more sense to write: $E_T = +\frac{\mu_0 q \omega^2 x_0}{4\pi} \frac{\sin\theta}{r} e^{i\omega\left(t - \frac{r}{c}\right)}$

Calculate the total power density) pick a phase, since we're about to time average anyhow

$$S = \frac{\mu_0 q^2 \omega^4 x_0^2}{16\pi^2 c} \frac{\sin^2\theta}{r^2} \cos^2\left\{\omega\left(t - \frac{r}{c}\right)\right\}$$

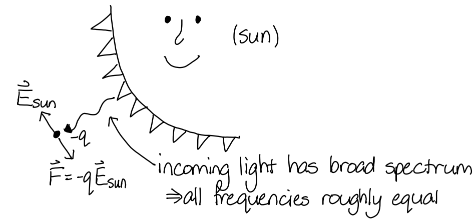
Time average: $\langle S \rangle = \frac{\mu_0 q^2 \omega^4 x_0^2}{32\pi^2 c} \frac{\sin^2\theta}{r^2}$

Total power radiation

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} \langle S \rangle r^2 \sin\theta \, d\theta = \frac{\mu_0 q^2 \omega^4 x_0^2}{16\pi c} \int_0^{\pi} \sin^3\theta \, d\theta$$

$\Rightarrow P = \frac{\mu_0 q^2 \omega^4 x_0^2}{12\pi c}$ increases with frequency as ω^4!

Rayleigh Scattering



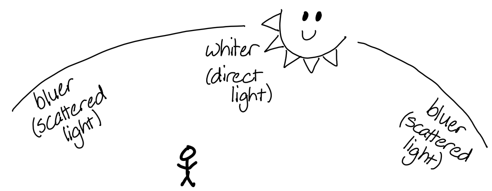
Molecules radiate power according to

$$P = \frac{\mu_0 q^2 \omega^4 x_0^2}{12\pi c}$$

⇒ more power is absorbed & re-radiated at high frequencies (i.e. blue)

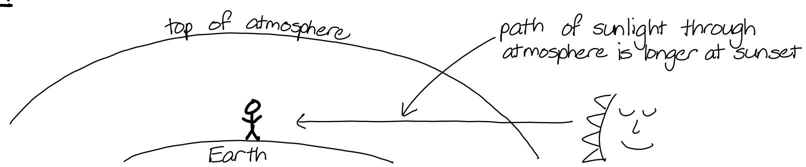
⇒ sky is BLUE

DAY



look up ⇒ sky is bluest farther from the sun

SUNSET



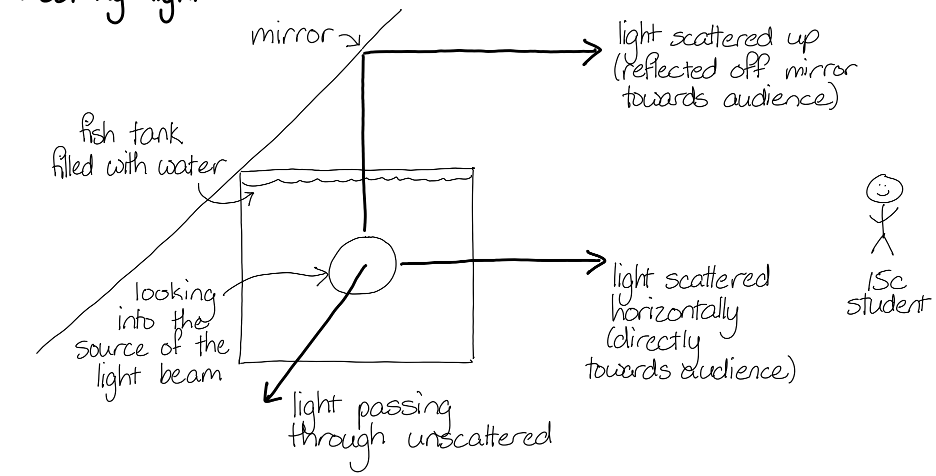
⇒ so much blue has scattered out along the longer path through the atmosphere that there's almost none left
⇒ area directly around sun no longer looks white
⇒ sunset looks red (= white - blue) instead

Beauty of our sky & sunset depends sensitively on the thickness of our atmosphere.
⇒ could be totally different on another planet

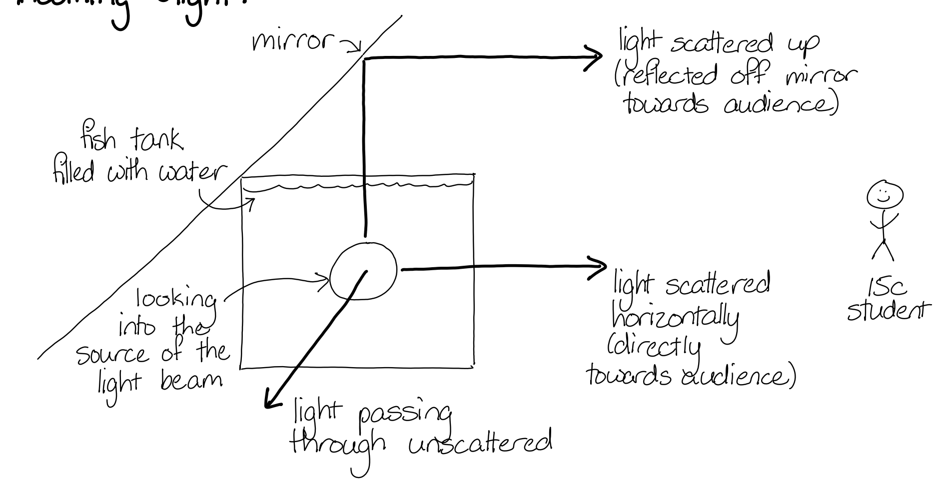
Test your conceptual understanding:

Demo: Polarization of Scattered Light

Vertically polarized incoming light:



Horizontally polarized incoming light:



Light in Matter: where does n come from?

n ← index of refraction

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

Let's write the current as: $\vec{J} = q n_0 \vec{v}$

- q ← charge of electron (take q to be negative)
- n_0 ← concentration of electrons
- $\vec{v}$ ← velocity of electron

Before we apply any fields, each electron is sitting in its own equilibrium position.

If we apply a field $\vec{E}$ (e.g. as from a wave passing through) then the electron will move in response to $\vec{E}$, but there will likely be some restoring force also wanting to pull it back to its original equilibrium position

Let's model the restoring force like a spring:

$$\vec{F}_{restore} = -k_s \vec{x}$$

k_s ← "spring constant"; $\vec{x}$ ← displacement of electron

Is this reasonable?

Here's another way to think of it: every electron is initially sitting in some potential well $U(x)$. The restoring force is therefore: $\vec{F}_{restore} = -\frac{\partial U}{\partial x}$

Remember way back in lecture #1 we realized that we can always linearize this for small displacements!
Do a Taylor expansion ⇒ everything behaves like a spring for small enough amplitudes.

If $\vec{E}$ changes slowly compared to the response time of the charge (i.e. frequency of light is slow compared to electron reaction) then at any given instant the electron is in equilibrium between the opposing forces from $\vec{E}$ and the restoring force.

So we can write $\Sigma \vec{F} = 0$: $\quad q\vec{E} - k_s\vec{x} = 0$

$$\Rightarrow \vec{v} = \frac{d\vec{x}}{dt} = \frac{q}{k_s}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{J} = q n_0 \vec{v} = \frac{q^2 n_0}{k_s}\frac{\partial \vec{E}}{\partial t}$$

Maxwell's eqns become:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \frac{\mu_0 q^2 n_0}{k_s}\frac{\partial \vec{E}}{\partial t}$$

combine these terms

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\varepsilon_0 + \frac{q^2 n_0}{k_s}\right)\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \text{define } \varepsilon \equiv \varepsilon_0 + \frac{q^2 n_0}{k_s}$$

Then we can write: $\vec{\nabla}\times\vec{B} = \mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t}$

We just absorbed $\vec{J}$ into matter's permittivity.

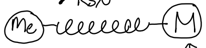
Then $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1 + \frac{q^2 n_0}{\varepsilon_0 k_s}}$

Great, now we know how to calculate n... but what is k_s?

Insulators (e.g. air)

e^- are bound to molecules

$qE \leftarrow \quad \rightarrow k_s x$



↑ ion is so much heavier that we ignore its movement

$$m_e \ddot{\vec{x}} = q\vec{E}_0 e^{-i\omega t} - k_s \vec{x}$$

Equation of motion for forced oscillation!
We have solved this before...

Plug in $\vec{x}(t) = \vec{x}_0 e^{-i\omega t}$

$$\Rightarrow -m_e\omega^2\vec{x}_0 = q\vec{E}_0 - k_s\vec{x}_0$$

$$\Rightarrow \vec{x}_0 = \frac{q\vec{E}_0}{k_s - m\omega^2} = \frac{q\vec{E}_0}{m_e(\omega_0^2-\omega^2)} \qquad \text{where } \omega_0 = \sqrt{\frac{k_s}{m_e}}$$

$$\Rightarrow \vec{J} = qn_0\vec{v} = \frac{-i\omega q^2 n_0}{m_e(\omega_0^2-\omega^2)}\vec{E}_0 e^{-i\omega t}$$

Wave equation is:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \mu_0\frac{\partial\vec{J}}{\partial t}$$

Comes from:

① $\frac{\partial}{\partial t}\left[\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}\right]$

② $\vec{\nabla}\times\left[\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right]$

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$\Rightarrow -k^2\vec{E} = \frac{-\omega^2}{c^2}\vec{E} - \frac{\mu_0 q^2 n_0}{m_e}\frac{\omega^2}{\omega_0^2-\omega^2}\vec{E}$$

Dispersion relation is:

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{q^2 n_0}{\varepsilon_0 m_e(\omega_0^2-\omega^2)}\right) = \frac{\omega^2}{c^2}\left(1 + \rho\frac{\omega_0^2}{\omega_0^2-\omega^2}\right)$$

where $\rho \equiv \frac{q^2 n_0}{\varepsilon_0 k_s}$ (unitless)

Air is apparently dispersive:

$$c_{phase} = \frac{\omega}{k} = \frac{c}{\sqrt{1+\rho\omega_0^2/(\omega_0^2-\omega^2)}}$$

$$\Rightarrow n = \sqrt{1 + \frac{\rho\omega_0^2}{\omega_0^2-\omega^2}}$$

Air - is a mixture of N_2, O_2, H_2O, Ar, etc.
→ many resonances exist at UV and shorter wavelengths

Luckily, things are simpler in visible light

$$\omega_0 \gg \omega$$

ω_0: frequency of resonances (UV, etc.)
ω: frequency of visible light

$$c_{phase} = \frac{c}{\sqrt{1+\rho\omega_0^2/(\omega_0^2-\omega^2)}} \approx \frac{c}{\sqrt{1+\rho}}$$

$$\Rightarrow n \approx \sqrt{1+\rho} \approx 1 + \tfrac{1}{2}\rho \qquad \left(\text{recall } \rho = \frac{q^2 n_0}{\varepsilon_0 k_s}\right)$$

At STP, $n = 1.0003 \Rightarrow \rho = 0.0006$

but ρ is proportional to density n_0
→ use ideal gas law

$$PV = n_{mol}RT$$

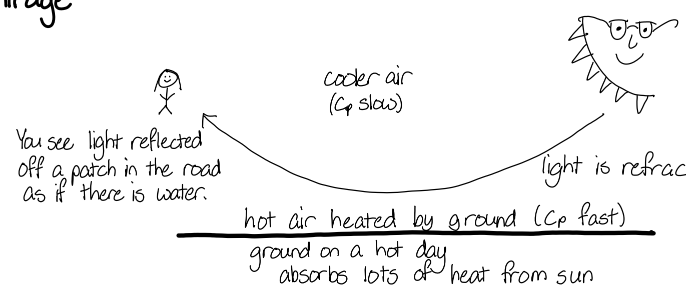
n_{mol}: # of moles; R: constant 8.314 J/mol·K

$$\text{density } n_0 \propto \frac{n_{mol}}{V} = \frac{P}{RT}$$

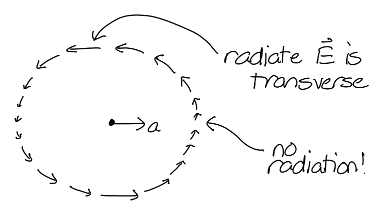
⇒ ρ goes ↑ as P goes ↑
but goes ↓ as T goes ↑

$$n = 1 + 0.0003\left(\frac{273}{T(K)}\right)\left(\frac{P(atm)}{1}\right) = \text{index of refraction for low-frequency (optical) EM waves}$$

Mirage



Summary



* EM radiation:

$$E \approx \frac{\mu_0 q}{4\pi}\frac{\sin\theta}{r}a\left(t - \frac{r}{c}\right)$$

at large r

* Larmor formula: $P = \frac{\mu_0 q^2 a^2}{6\pi c}$

* Oscillating charge: $P = \frac{\mu_0 q^2\omega^4 x_0^2}{12\pi c}$

* Rayleigh scattering: sky is blue, sunset is red

* Index of refraction: $n = \sqrt{1 + \frac{\rho\omega_0^2}{\omega_0^2-\omega^2}}$ where ω_0 = resonance of molecules