Physics 15c (Hoffman)
Lecture # 8
Tues, Sept 28, 2010

Wrap-up & Review

Focus on:
* Fourier transforms
  example: derivation of group velocity
* When do we get to drop the imaginary part?
* Symmetry matrices

① Discrete:

We can write any periodic $f(t)$ with period T as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right\}$$

where a_n and b_n are given by the "inner product":

$$a_n = \frac{1}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right)dt$$

② Continuous:

$$f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) \underbrace{e^{-i\omega t}}_{\text{basis vectors}} d\omega$$

coefficient of linear combo of basis vectors

need to integrate instead of add to find linear combo, because ω is continuous

Now $\tilde{F}(\omega)$ comes from an "inner product" of $f(t)$ with basis state:

$$\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

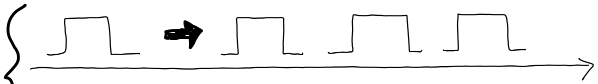
$$\boxed{f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega t} d\omega}$$ ← inverse Fourier transform

$$\boxed{\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt}$$ ← Fourier transform

How does a real pulse disperse?
* derivation of $v_{group} = \frac{d\omega}{dk}$
* practice with Fourier transform

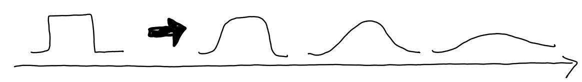
Imagine a pulse being sent over a distance:

non-dispersive



* on a non-dispersive medium, pulse shape is unchanged
* because all normal modes (all the sines & cosines that add together to give the pulse) have the same v_{phase}

dispersive



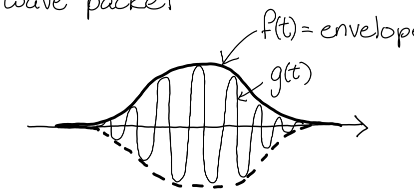
* on a dispersive medium, the pulse shape must change
* pulse shape gets "dispersed"

→ dispersion makes a poor medium for communication!

Wave packets

Information is carried by wave packets
→ consider a general wave packet

$$g(t) = \underbrace{f(t)}_{\text{pulse shape}} \underbrace{e^{-i\omega_c t}}_{\text{carrier wave}}$$



→ modulate the carrier wave $e^{-i\omega_c t}$ with a pulse $f(t)$

Now we're going to use Fourier analysis to understand how this packet propagates. This will lead naturally to a derivation of $v_{group} = \frac{d\omega}{dk}$ (recall: last time we sort of jusified it using just 2 waves & looking at the velocity of their beats)

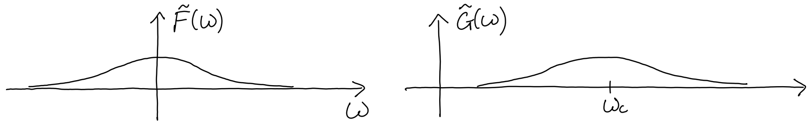
Fourier integral of the wave packet is:

$$\tilde{G}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega_c t}e^{i\omega t}dt$$

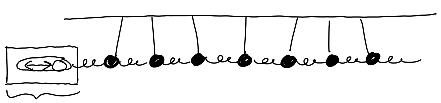
$$= \tilde{F}(\omega-\omega_c) \quad \text{where} \quad \tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

The frequency spread of the wave packet $\tilde{G}(\omega)$ has the same shape as the frequency spread of just the pulse envelope $\tilde{F}(\omega)$, but $\tilde{G}(\omega)$ is centered at ω_c, while $\tilde{F}(\omega)$ is centered around 0



check: $\tilde{G}(\omega_c) = \tilde{F}(\omega_c-\omega_c) = \tilde{F}(0)$ ✓

Now we look at how the whole wave packet $g(t)$ propagates in space:



motor drives
$g(t) = f(t)e^{-i\omega_c t}$

Forward-going wave packet is generated at $x=0$ as:

$$\xi(0,t) = g(t) = \underbrace{\int_{-\infty}^{\infty}\tilde{G}(\omega)e^{-i\omega t}d\omega}_{\text{inverse Fourier transform}}$$

remember: $\tilde{G}(\omega) \neq 0$ only near $\omega = \omega_c$ (say within $\Delta\omega$ of ω_c)

→ we know how each individual normal mode travels

$$e^{-i\omega t} \text{ at } x=0 \rightarrow e^{i(kx-\omega t)}$$

→ so the total wave packet should travel as the sum (integral)

$$\xi(x,t) = \int_{-\infty}^{\infty}\tilde{G}(\omega)e^{i(kx-\omega t)}d\omega$$

where this k is $k(\omega)$ from the dispersion relation

$$\xi(x,t) = \int_{-\infty}^{\infty}\tilde{G}(\omega)e^{i[k(\omega)x-\omega t]}d\omega$$

$$= \int_{\omega_c-\Delta\omega}^{\omega_c+\Delta\omega}\tilde{G}(\omega)e^{i[k(\omega)x-\omega t]}d\omega$$

(where $2\Delta\omega$ is the full width of the frequency range contained in this particular pulse.)

$$= \int_{\omega_c-\Delta\omega}^{\omega_c+\Delta\omega}\tilde{F}(\omega-\omega_c)e^{i[k(\omega)x-\omega t]}d\omega$$

Taylor expand: $k(\omega) = k(\omega_c) + \left.\frac{dk}{d\omega}\right|_{\omega_c}(\omega-\omega_c) + \frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}(\omega-\omega_c)^2 + \ldots$

change variables: $\omega' = \omega-\omega_c$

$$= \int_{-\Delta\omega}^{\Delta\omega}\tilde{F}(\omega')e^{i\left[\left(k_c+\omega'\left.\frac{dk}{d\omega}\right|_{\omega_c}+\frac{1}{2}(\omega')^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}\right)x-(\omega_c+\omega')t\right]}d\omega'$$

$$= e^{i(k_c x-\omega_c t)}\int_{-\Delta\omega}^{\Delta\omega}\tilde{F}(\omega')e^{i\omega'\left[\left.\frac{dk}{d\omega}\right|_{\omega_c}x-t\right]}\underbrace{e^{\frac{i}{2}(\omega')^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}x}}_{\text{This is the 2nd order Taylor term, presumably small, so let's ignore it for now}}d\omega'$$

$$\xi(x,t) = e^{i(kx-\omega_c t)} \underbrace{\int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{-i\omega'\left(t - \frac{dk}{d\omega}\big|_{\omega_c} x\right)} d\omega'}$$

This part looks like an inverse Fourier transform except that t is shifted by $\frac{dk}{d\omega}\big|_{\omega_c} x$

$$\xi(x,t) = \underbrace{e^{i(k_c x - \omega_c t)}}_{\text{carrier wave}} \underbrace{f\left(t - \frac{dk}{d\omega}\Big|_{\omega_c} x\right)}_{\text{shape of the wave packet}}$$

What did we expect to find?

For a non-dispersive medium, we just expect that the shape $f(t)$ propagates with some velocity v_{group}, so that at any time & position, it would look like $f(t - x/v_{group})$

Ah-hah! So we can identify $\frac{1}{v_{group}} = \frac{dk}{d\omega}\Big|_{\omega_c}$

$$\Rightarrow \boxed{v_{group} = \frac{d\omega}{dk}\Big|_{\omega_c}} \leftarrow \textbf{derivation of } v_{group}$$

Using only the first term of the Taylor expansion in the exponent, we approximated this as a dispersionless wave.

For a dispersive medium, things get more algebraically involved...

$$\xi(x,t) = e^{i(k_c x - \omega_c t)} \int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{i\omega'\left[\frac{dk}{d\omega}\big|_{\omega_c} x - t\right]} \underbrace{e^{\frac{i}{2}(\omega')^2 \frac{d^2k}{d\omega^2}\big|_{\omega_c} x}}_{\text{keep this term too now}} d\omega'$$

But everything in this messy expression is, in principle, known. Presumably, you know the wave eqn and therefore the dispersion relation for your medium, so you can compute $\frac{dk}{d\omega}$ and $\frac{d^2k}{d\omega^2}$

And presumably you know the shape of the pulse you started with, $g(t)$, so you can ask Mathematica to calculate:

$$\tilde{F}(\omega') = \tilde{F}(\omega - \omega_c) = \tilde{G}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t) e^{i\omega t} dt$$

Then you can put all the pieces together and ask Mathematica to calculate:

$$\xi(x,t) = e^{i(k_c x - \omega_c t)} \int_{-\Delta\omega}^{\Delta\omega} \tilde{F}(\omega') e^{i\omega'\left[\frac{dk}{d\omega}\big|_{\omega_c} x - t\right]} e^{\frac{i}{2}(\omega')^2 \frac{d^2k}{d\omega^2}\big|_{\omega_c} x} d\omega'$$

So taking a step back, this is pretty sweet:

We started knowing just the dispersion relation of our medium, $\omega(k)$, and the shape of our pulse $g(t)$ at the origin $x=0$.
By breaking it into a sum of normal modes (i.e. Fourier transforming it) we were able to:

1. derive the group velocity $v_{group} = \frac{d\omega}{dk}$
2. write an expression for $\xi(x,t)$ = the evolution of $g(t)$ across all space, for all times i.e. we wrote down an analytic formula for how the pulse shape changes as it propagates

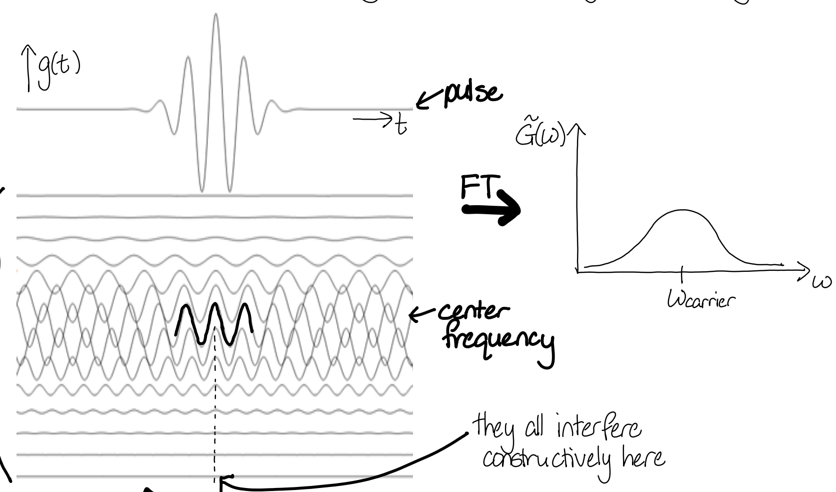
Summary

Used Fourier analysis to derive $v_{group} = \frac{d\omega}{dk}$

and to compute analytically how a pulse spreads as it propagates, knowing only the dispersion relation and the initial pulse shape

Q: Still not really sure how Fourier transforms work.

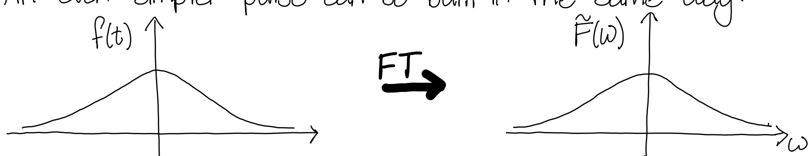
A pulse can be built out of many waves of nearby frequency:



These waves of varying amplitudes & frequencies sum to make whole pulse.

but farther away from the central time, the components are getting farther and farther out of phase, and eventually interfere totally destructively

An even simpler pulse can be built in the same way:



This is symmetric about $t=0$ so we know it is composed of a continuous integral of cosines but $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

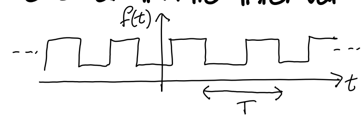
so for every $+\omega$ component there should be an equal amplitude $-\omega$ component

$\Rightarrow$ $\tilde{F}(\omega)$ is also symmetric about $\omega = 0$

Q: Can Fourier series be useful to represent non periodic functions over given intervals?

This is a subtle question. Over an infinite interval (i.e. $f(t)$ defined for all time $-\infty < t < \infty$), a continuous Fourier integral is required to represent an arbitrary non-periodic

Over an infinite interval:



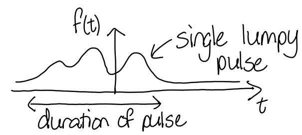
If the whole function is periodic, with period T, then you'll never need anything that isn't periodic in T to compose it. In fact, if you try to throw in a component that isn't periodic in T, you'll just mess it up.

So you can write it as a series of all functions that are periodic in T, in 2 different, equivalent ways:

① $f(t) = \sum_{n=0}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right\}$

OR

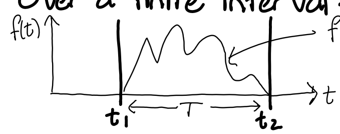
② $f(t) = \sum_{n=0}^{\infty} \left\{ c_n e^{i\left(\frac{2\pi n t}{T}\right)} + d_n e^{-i\left(\frac{2\pi n t}{T}\right)} \right\}$



If the whole function is not periodic then you're going to need arbitrarily finely spaced frequencies to ensure complete destructive interference at all times outside the pulse duration.

$$f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{i\omega t}\, d\omega$$

Over a finite interval:



function $f(t)$ only defined for $t_1 < t < t_2$

Now, since $f(t)$ is not defined outside of $t_1 < t < t_2$, we can just pretend that it repeats forever outside that interval, with period T. So we can write

$$f(t) = \sum_{n=0}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right\}$$

where $T = t_2 - t_1$

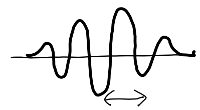
Q: Confused about the relation between v_group and v_phase.
Q: Group vs. carrier frequencies.

v_{group} = velocity of the pulse

But the pulse can be decomposed into a sum (integral) of many pure sines and cosines at different frequencies.

$v_{phase}(\omega)$ = velocity of the component (pure sine or cosine) at a particular frequency ω.

Often a pulse will have a form like this:



i.e. a fast frequency (carrier) with a slower envelope of amplitude.

$T_{carrier} = \frac{2\pi}{\omega_{carrier}}$

Then the Fourier spectrum will be peaked at $\omega_{carrier}$.

$v_{carrier} = v_{phase}(\omega_{carrier})$

To compute any of these, start from the dispersion relation $\omega(k)$:

$v_{group} = \frac{d\omega}{dk}$ $\quad v_{phase} = \frac{\omega}{k}$

Q: How to get a dispersion relation from a harmonic wave when you're not sure what's defined as k and w. Such as problem 5 on homework 3.

The dispersion relation is usually written as an equation relating ω and k. But in a simple system where all solutions manifestly have the same velocity [i.e. $f(x-vt)$] then you can write:

$$\text{velocity} = \frac{dx}{dt} = \frac{\omega}{k} \Rightarrow \omega = \underbrace{\left(\frac{dx}{dt}\right)}_{\text{i.e. compute this from your functional form and you are done.}} k$$

Q: Is there is a quantity which is the "dispersion"?

The dispersion is not a single quantity, but rather an equation which relates ω to k. This equation is a property of a given medium, i.e. water has a dispersion, a slinky has a different dispersion, etc.

Q: When you're supposed to take the real part and when not.
Q: Imaginary and real solutions. What's possible and what is not?
Q: The whole deal with real and imaginary components is still a little fuzzy.

Ok, let's be totally clear up front: the **only** time you are ever "allowed" to just throw away the imaginary part of a solution is when you've explicitly created a situation in which the whole imaginary part of the equation is bogus. In this class, we did this only once, with the LCR circuit (and the analogous geophone problem on the homework).

In those cases, we started with a real equation for a physical system: $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f(t)$

where all pieces are real. Then we intentionally added in some bogus imaginary stuff on the right, so we had to add some bogus stuff to $x(t) \rightarrow z(t) = x(t) + \underbrace{i x_1(t)}_{\text{bogus stuff!}}$

At the end of the process, we found a **single** function for $z(t)$ and we threw away the bogus stuff so we were left with a **single** real $x(t)$ which turns out to be the (unique) steady state solution.

In **all other** examples, we have likewise started with a real equation for a physical system, such as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{OR} \quad m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & K & \cdots \\ & \vdots & \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

$$\text{OR} \quad \frac{\partial^2 \xi}{\partial t^2} = c^2\frac{\partial^2 \xi}{\partial x^2} + \omega_0^2 \xi$$

but in all of these cases, we did not add any bogus stuff.

We just solved the equations as-is, and we typically found **multiple** solutions for $x(t)$ or $\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ or $\xi(x,t)$ which were complex.

These are the true solutions to these physical equations. There is nothing bogus about this imaginary stuff.

BUT: the math saves us from the imaginary solutions after all. Because the original equations are all real, their solutions must be real, or must come in complex conjugate pairs. The total solution is a linear combination of these **multiple** solutions. In particular, for any given pair $x(t)$ and $x^*(t)$ we can just rewrite as different linear combinations:

$$x_1(t) = x(t) + x^*(t) \equiv Re(x(t))$$

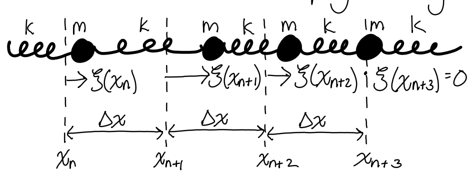
$$x_2(t) = -i(x(t) - x^*(t)) \equiv Im(x(t))$$

So you see we can get the real and imaginary parts legitimately, as linear combinations, without just throwing anything away!

Q: Still not sure how to find the elastic modulus.

Let's take a broader approach to answering this, and re-investigate the transition from N masses to the continuum limit.

We start with this discrete physical system:



We take a "continuum" limit as $\Delta x \to 0$, and arrive at the wave eqn:

$$\boxed{\frac{m}{\Delta x}\frac{d^2}{dt^2}\xi(x) = k\Delta x \frac{d^2\xi}{dx^2}}$$

In order for this system to remain physical, though, we need to think carefully about what happens to m and k as $\Delta x \to 0$.

If m were to stay constant as $\Delta x \to 0$, that would mean that an infinite number of finite masses are coming infinitesimally close together → in other words we created a **BLACK HOLE**

Obviously, that's bad.

So the sensible thing is to require that $m \to 0$ as $\Delta x \to 0$ and in particular it's convenient to just fix the linear mass density: $\rho_L = \frac{m}{\Delta x}$ = constant as $\Delta x \to 0$

Now, what about k? According to our picture, Δx is the equilibrium length of each spring separating 2 masses, and the displacement from equilibrium (for the n^{th} pair of masses) is $x_n - x_{n-1}$. Let's rename these:

$\Delta x \to L$ = equilibrium length of spring
$x_{n+1} - x_n \to \Delta L$ = displacement from equilibrium length

According to Hooke's law, k is the constant of proportionality between the displacement ΔL and the resultant force F.

$$F = -k\Delta L \qquad (1)$$

But this k is an extrinsic property of the spring. As we make Δx (or L) shorter, for the same spring material, the value of k is going to blow up. So we're going to need to find some property of the spring which is intrinsic, which stays the same as $\Delta x \to 0$ (just as ρ_L stays the same as $\Delta x \to 0$).

So let's define a new "elastic constant" E which will allow us to rewrite Hooke's law in terms of the fractional change in length of the spring rather than the absolute change in length.

$$F = -E(\Delta L/L) \qquad (2)$$

By comparing (2) to (1) we can see that $E = kL$

Now, returning to the terminology of our mass-spring system:

$$\Delta L \to x_{n+1} - x_n = \xi(x+\Delta x) - \xi(x)$$
$$L \to \Delta x$$
$$E \to k\Delta x$$
$$F = -E(\Delta L/L) \to F = -(k\Delta x)\frac{\xi(x+\Delta x) - \xi(x)}{\Delta x}$$