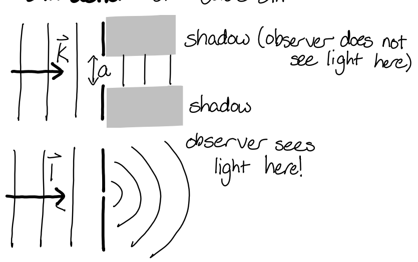
Physics 15c (Hoffman)
Lecture #21
Tues, Nov 23, 2010

# Introduction to Lenses

## Last time:

**Diffraction** of wide slit:

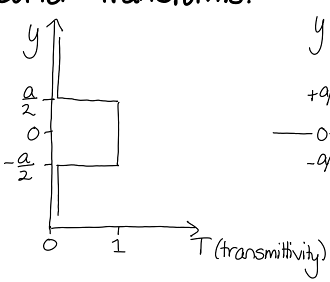


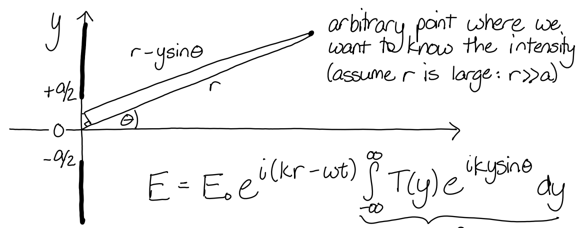

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin x}{x}\right)^2$$

where $x \equiv \frac{ka\sin\theta}{2} = \frac{\pi a \sin\theta}{\lambda}$

⇒ we see a broad central maximum whose angular width is inversely proportional to the slit width

## Fourier Transforms:



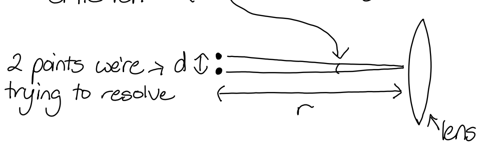




$$E = E_0 e^{i(kr-\omega t)} \underbrace{\int_{-\infty}^{\infty} T(y) e^{iky\sin\theta}\, dy}_{\text{Fourier transform between variables } y \longleftrightarrow k\sin\theta}$$

## Optical resolution

The resolution of an optical device with aperture $a$ is:

"Rayleigh criterion" $\left\{ \sin\theta > 1.22\frac{\lambda}{a} \right.$




Note: $\sin\theta \approx \frac{d}{r}$

so if we make $d$ smaller <u>or</u> $r$ bigger, then we run into trouble

**Interference** vs. **Diffraction** → what is the difference?

Nothing; diffraction is just the limiting case of interference for a continuous distribution of point sources (i.e. wide slit)



## Goals for today

* Intro to lenses (terminology)
* Spherical aberration
* Chromatic aberration
* Rainbow

**Next time:** more quantitative look at specific optical devices: magnifying glass, microscope, etc.

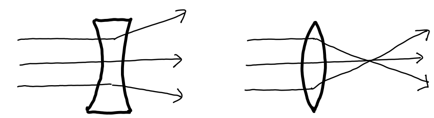
**Geometrical Optics** = analysis of optical devices using the following assumptions

* assume all elements (lenses, mirrors, etc) are much larger than wavelength $\lambda$, i.e.:
  - $a \gg \lambda$ (aperture ≫ wavelength)
  - $t \gg \lambda$ (thickness ≫ wavelength)
* treat light as if it's a particle: straight line trajectory in each medium
* at boundaries, light either reflects or refracts (refraction angles given by Snell's law)

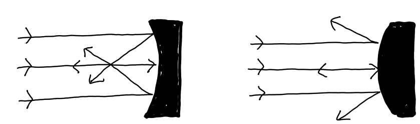
⇒ Everything is fully determined by the elements' shapes, indices of refraction, and their geometrical arrangement

## 4 major types of optical elements:

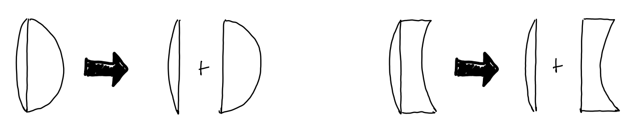
concave & convex lenses



concave & convex mirrors



Lenses may have different radii on 2 surfaces: consider them as a combination of 2 lenses with one flat side

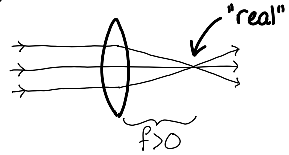


# Focal Point

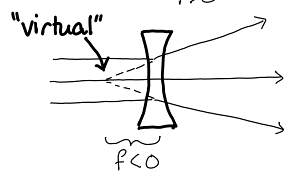
Lenses (& mirrors) turn plane waves into spherical waves:
"origin" of the spherical waves = focal point
Distance between lens and its focal point = focal length = $f$

Light may or may not actually go through the focal point

If light **does** go through the focal point → **real** focal point.
$\Rightarrow f > 0$




If light **does not** go through the focal point → **virtual** focal point.
$\Rightarrow f < 0$




# Diopter

Optometrist prescribes glasses using diopter

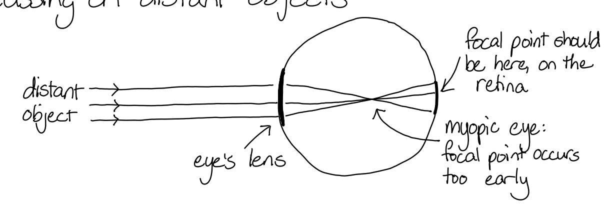
$\text{diopter} = \frac{1}{f}$ = "optical power"

Larger number → shorter focal length → stronger bending of light

# Quiz

**(a)** What is the focal length of a $-1.5$ diopter lens?
Is it concave or convex?

**(b)** Myopic means "near-sighted" which means that the eye has trouble focussing on distant objects

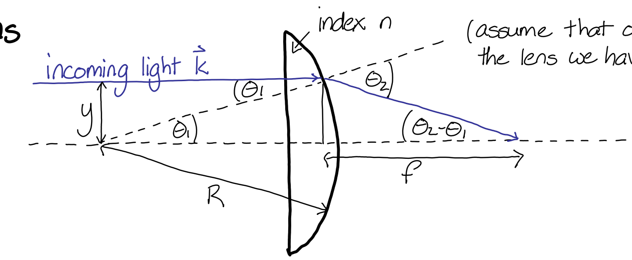



Question: do you need a concave or convex lens to correct myopic vision?



# Convex lens

(assume that outside the lens we have $n_{air} \approx 1$)




Convex side is spherical, so the radius is $\perp$ to surface.
Use Snell's law: $n\sin\theta_1 = \sin\theta_2$

$$\Rightarrow \sin\theta_2 = \frac{ny}{R}$$

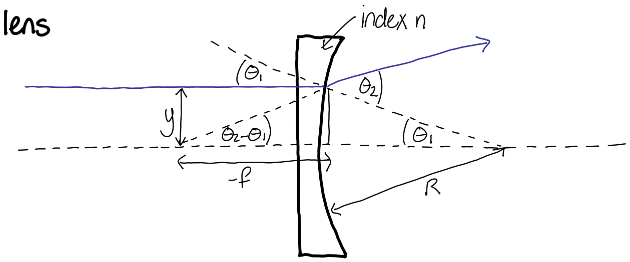
For small angles, the focal length is:

$$f = \frac{y}{\tan(\theta_2-\theta_1)} \approx \frac{y}{\theta_2-\theta_1} \approx \frac{y}{\frac{ny}{R}-\frac{y}{R}} = \frac{R}{n-1}$$

i.e. the focal length is independent of height $y$ of incoming light

Incoming light converges at the focal point, but there are approximations!

# Concave lens




Concave side is spherical, so the radius is $\perp$ to surface.
Use Snell's law: $n\sin\theta_1 = \sin\theta_2$

$$\Rightarrow \sin\theta_2 = \frac{ny}{R}$$

For small angles, the focal length is:

$$-f = \frac{y}{\tan(\theta_2-\theta_1)} \approx \frac{y}{\theta_2-\theta_1} \approx \frac{y}{\frac{ny}{R}-\frac{y}{R}} = \frac{R}{n-1}$$

Same formula, just a negative sign.
Again, we see that focal length is independent of height of incoming light.
But don't forget, there are approximations!

## Aberrations

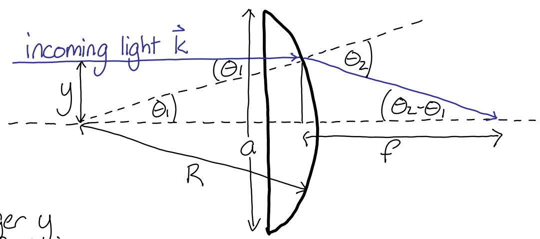
Two approximations were made:

1) angles $\theta_1$ and $\theta_2$ are small
2) index $n$ is constant for all $\lambda$

Both are incorrect for real lenses

1) Angles may get large if the aperture is large
   → rays at different $y$ do not all converge at the same $f$
   → **spherical aberration**
2) Index varies with wavelength $\lambda$ due to dispersion
   → rays with different $\lambda$ do not converge at the same $f$
   → **chromatic aberration**

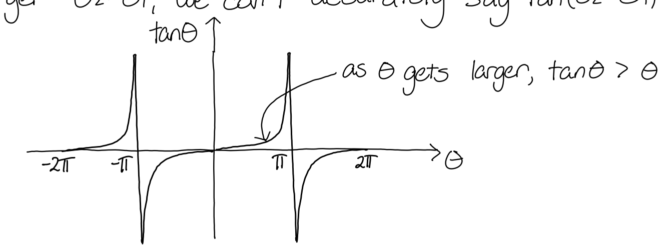
## Spherical aberration




Incoming light farther from the lens center (i.e. larger $y$) makes a bigger angle $\theta_1$ with $R$.
Snell: $n\sin\theta_1 = \sin\theta_2$
⇒ $\theta_2$ is bigger too for bigger $y$
⇒ $\theta_2-\theta_1$ is bigger too for bigger $y$

From geometry, we still have $f = \dfrac{y}{\tan(\theta_2-\theta_1)}$

but for larger $\theta_2-\theta_1$, we can't accurately say $\tan(\theta_2-\theta_1) \approx \theta_2-\theta_1$




So for larger $y$, $\tan(\theta_2-\theta_1) > \theta_2-\theta_1$, so $f = \dfrac{y}{\tan(\theta_2-\theta_1)} <$ expected

So rays passing near the perimeter of the lens "over-refract", i.e. they are bent further down, to smaller $f$ than expected.
This problem is negligible for small aperture $a \ll R$.



## F-stop

Camera lens "f-stop" $\equiv \dfrac{f}{a} = \dfrac{R}{a(n-1)}$

[Weird notation: "f-stop" is denoted as "f/#" where $\# = \dfrac{f}{a} = \dfrac{R}{a(n-1)}$]

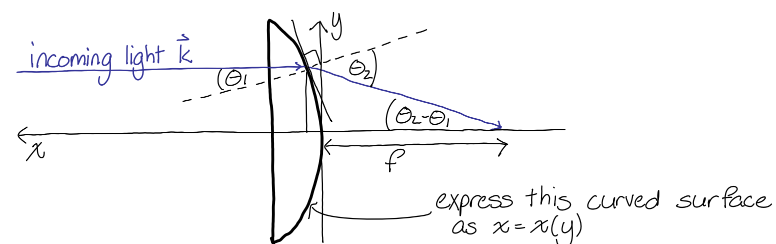
Smaller f-stop = larger aperture = brighter (faster) lens
But camera lenses with small f-stop suffer from more spherical aberration.
Good 50mm lenses have f/1.4 or smaller → i.e. $\frac{R}{a(n-1)} = 1.4$
e.g. Canon EL 50mm lens has f/1.0 (i.e. $\frac{R}{a(n-1)} = 1.0$ ⇒ $a$ is big ⇒ fast lens!)

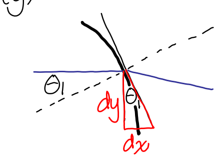
How do they get around the spherical aberration problem?

1) Use glass with high index of refraction
   Recall $f = \dfrac{R}{n-1}$ so larger $n$ makes $R$ larger for same $f$
   → smaller aberration for the same $a$
   Flint glass has $n = 1.575 - 1.89$
   (compare to normal glass: $n = 1.5$)
2) Non-spherical lens

## Aspherical lens

Can we find a specific shape for a lens that will eliminate spherical aberration?




① slope at height $y$ satisfies $\tan\theta_1 = \dfrac{dx}{dy}$

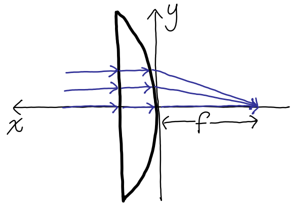
② Snell's law: $n\sin\theta_1 = \sin\theta_2$

③ focal length is $f = \dfrac{y}{\tan(\theta_2-\theta_1)} - x(y)$

Three eqns, three unknowns $(x, \theta_1, \theta_2)$ → can solve for $x(y)$
But pretty ugly → let's look for a trick.

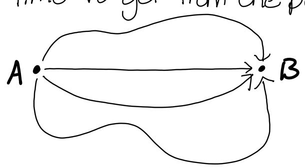
## Fermat's Principle

Consider incoming plane waves
→ all rays have the same phase as the enter the lens

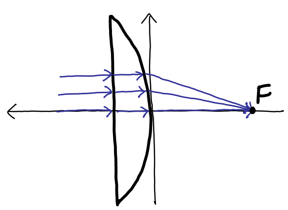



When the rays meet at the focal point, they must have the same phase, otherwise they won't look like spherical waves.
In order to have the same phase at the focal point, all rays must take the same amount of time to get there.

In more general form, Fermat's principle says "light takes the shortest amount of time to get from one point to another."




Light could take any crazy path from A to B.
But arriving light at B from all those different paths would have all different phases, and would mostly interfere destructively.
But, the extremum path (the shortest path) will have an extremum in phase too, i.e. if $\beta$ is some parameter to parametrize all paths then $d\varphi/d\beta$ will be zero for the direct path.
⇒ nearby paths will have nearly the same phase as the direct path
⇒ these paths around the direct path will interfere constructively while all other paths are interfering destructively
⇒ it's as if the light just takes the direct path of least time



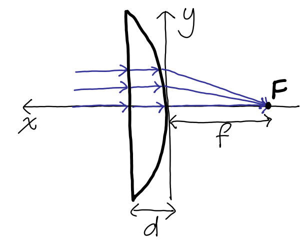

Among all paths to F, only the shortest one should actually be taken.
For all the rays to reach exactly F (i.e. no spherical aberration) they must all be "the shortest" i.e. they must all take equal time.

Difference in geometrical lengths is compensated by the slowdown due to n inside the lens.



Time from the flat surface to F is:

$$t(y) = \frac{n}{c}(d - x(y)) + \frac{1}{c}\sqrt{(f + x(y))^2 + y^2}$$




If this is to be constant, then $t(y) = t(0)$ for any arbitrary value of y.

$$\frac{n}{c}(d - x(y)) + \frac{1}{c}\sqrt{(f + x(y))^2 + y^2} = \frac{n}{c}d + \frac{1}{c}f$$

Solve for $x(y)$:

$$f + n\,x(y) = \sqrt{(f + x(y))^2 + y^2}$$

$$(f + n\,x(y))^2 = (f + x(y))^2 + y^2$$

$$\Rightarrow (n^2-1)x^2 + 2(n-1)fx - y^2 = 0 \quad \Big\}\ \text{hyperbola}$$

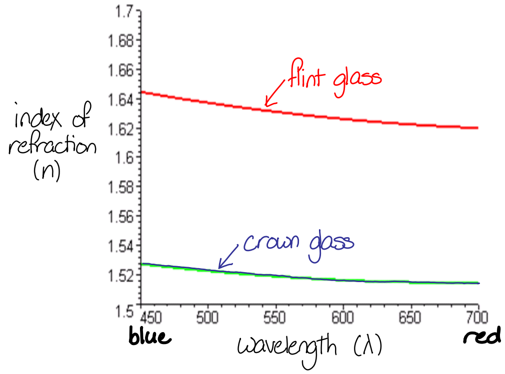
Note: this is not a perfect solution for several reasons
1) doesn't work for off-axis light
2) difficult to make with traditional polishing technique

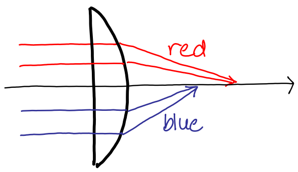
In summary, spherical aberration can be reduced by
1) high-index glass (flint glass)
2) aspherical (hyperbolic) lens shape
3) combining multiple lenses so that aberrations cancel

Designing good lenses remains on borderline btwn art & science.
(Many photographers still believe 60-year-old Zeiss lenses are better than modern computer-designed ones.)

## Chromatic aberration

Index of refraction varies with wavelength
$n_{blue} > n_{red}$

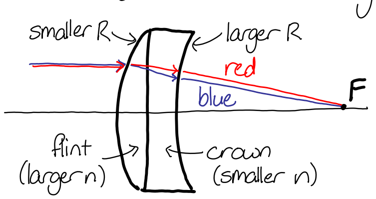



Recall formula: $f = \frac{R}{n-1}$

→ shorter f for blue than for red
→ blue light bends more than red light

How can we get rid of chromatic aberration?
Idea: use 2 different kinds of glass with different dispersion




Combine concave & convex lens so dispersion cancels out
→ **achromatic lens**

Flint glass was created in early 18th century
Achromatic lens was invented by C.M. Hall in 1730s
(he kept it a secret, ordering the two lenses from different lens companies)
G. Bass, a subcontractor of the two lens companies realized the secret, but also kept it to himself.
Idea was leaked to J. Dolland, who re-invented the achromatic lens in 1759, got patent, made big money.

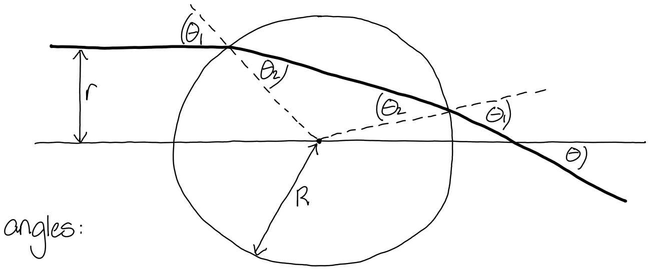


## Glass Sphere

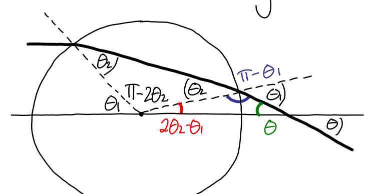
A glass ball is the worst case for spherical aberration

① $\sin\theta_1 = \frac{r}{R}$

② $\sin\theta_1 = n\sin\theta_2$



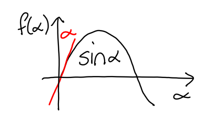

Fill in some more angles:




⇒ Total bending angle is θ.

$$\theta = \pi - (\pi - \theta_1) - (2\theta_2 - \theta_1) = 2(\theta_1 - \theta_2)$$ } this is exact

$$= 2\left[\sin^{-1}\left(\frac{r}{R}\right) - \sin^{-1}\left(\frac{r}{nR}\right)\right]$$

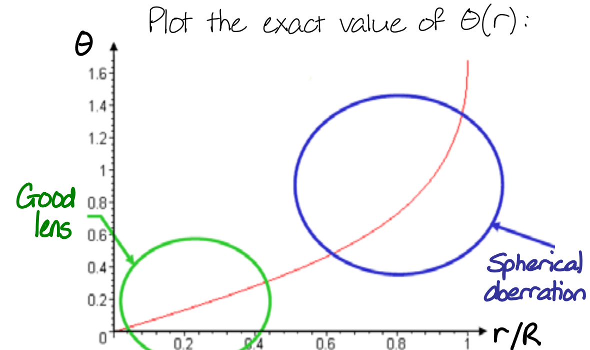



For small $\alpha$, $f(\alpha) = \sin\alpha \approx \alpha$
For larger $\alpha$, $f(\alpha) = \sin\alpha < \alpha$
$\Rightarrow \sin^{-1}[f(\alpha)] < \sin^{-1}(\alpha)$
$\Rightarrow \alpha < \sin^{-1}(\alpha)$

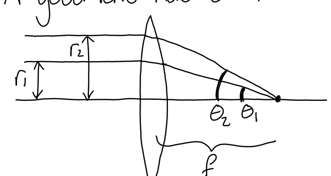
⇒ For small r, $\theta \approx 2\left(\frac{r}{R} - \frac{r}{nR}\right) = \frac{2(n-1)r}{nR}$

But for larger r, $\frac{2(n-1)r}{nR} < \theta$

Plot the exact value of θ(r):




Remember:
A good lens has $\theta \propto r$:




In order to converge at the same focus f, a good lens has

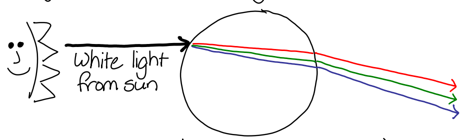
$f = \frac{r_1}{\tan\theta_1} = \frac{r_2}{\tan\theta_2}$

$\Rightarrow r_1/\theta_1 \approx r_2/\theta_2$

$\Rightarrow r \propto \theta$

# Rainbow

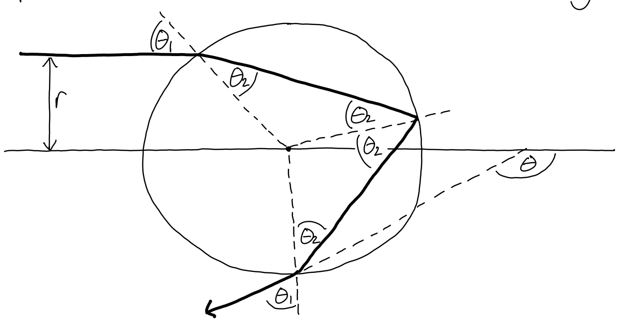
Over-simplified (wrong!) picture that you probably have vaguely in your head:



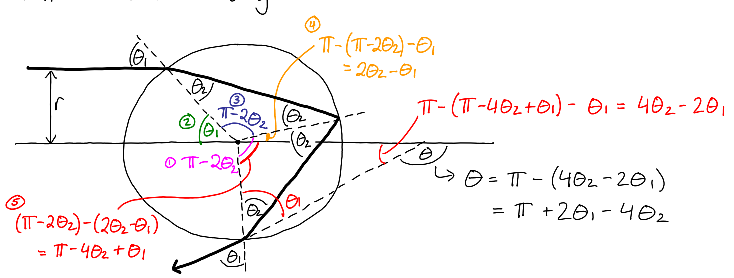

raindrop acts as a prism to separate the white light into colors

But the problem with this simplified view is that the scattering angle depends on where the light enters the drop.
So for a given color, it won't be focused at a single place.
If you add up all the possible positions, the rainbow will be washed out.

Correct picture of a rainbow comes from internally reflected light:




Fill in some more angles:




$\hookrightarrow \theta = \pi-(4\theta_2-2\theta_1)$
$= \pi+2\theta_1-4\theta_2$



Recall:

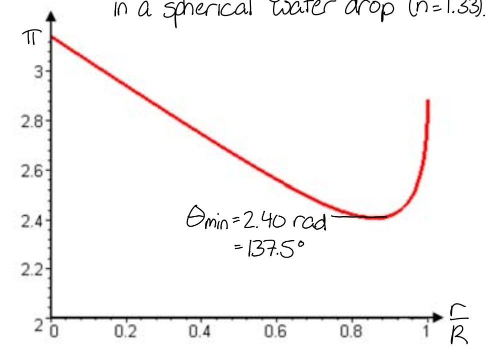
① $\sin\theta_1 = \frac{r}{R}$

② $\sin\theta_1 = n\sin\theta_2$

$\Rightarrow \theta = \pi + 2\theta_1 - 4\theta_2$

$= \pi + 2\sin^{-1}\left(\frac{r}{R}\right) - 4\sin^{-1}\left(\frac{r}{nR}\right)$

Plot $\theta(r)$ for internally reflected light in a spherical water drop ($n=1.33$).




For any old $r$, the output angle $\theta(r)$ is varying rapidly with $r$.
But for a range of $r$ around this special value, $\theta(r)$ is almost constant.

So for white light coming in at a range of $r$ values, there is a lot reflected at the particular value $\theta_{min}$. This light is not washed out - and thus becomes the rainbow.

Quantitatively, one can solve to find outgoing intensity at each $\theta$:
(a lot of algebra, not worth doing here)

$$I(\theta) \propto \frac{r}{\sin\theta}\underbrace{\left|\frac{dr}{d\theta}\right|}$$

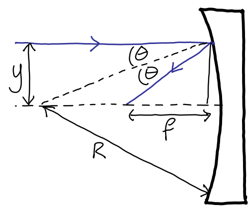
this part $\to \infty$ at $\theta_{min}$

but $\theta_{min}$ depends on $n$, which depends slightly on $\lambda$

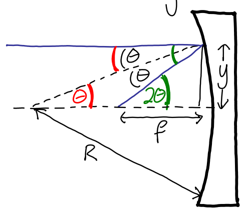
So each $\lambda$ peaks at a slightly different $\theta \Rightarrow$ RAINBOW

## Mirrors

Mirrors are simpler than lenses




Fill in additional angles:




$\Rightarrow y = f\tan(2\theta) = R\sin\theta$

For small angles: $f(2\theta) \approx R\theta \Rightarrow \boxed{f = R/2}$

* No chromatic aberration (n doesn't come into play at all)

* To avoid spherical aberration, need a parabolic mirror

## Telescopes

* Use a concave mirror instead of a convex lens (to focus)
  - → because it's easier to make a large mirror than a large lens
  - → can make the overall telescope length shorter

A bit of history:

Hubble space telescope launched in 1990 with spherical primary mirror.
- → spherical aberration made it nearly useless
- → corrective optics ("COSTAR") added in December 1993
  = "eyeglasses for Hubble"

## Summary

* Rayleigh criterion (Airy disc) ⇒ resolution limit: $\theta \approx \sin\theta > 1.22\frac{\lambda}{a}$
* Concave & convex lenses: $f = \frac{R}{n-1}$ $\quad$ ["f-stop" $= \frac{f}{a} = \frac{R}{(n-1)a}$
* Real vs. virtual images $\quad$ "diopter" $= \frac{1}{f} =$ "optical power"]
* Concave mirror: $f = R/2$
* Spherical aberration (corrected by hyperbolic lens, parabolic mirror)
* Fermat's principle: light takest shortest time from A to B
* Rainbow: internal reflection from water sphere