Physics 15b (Hoffman)
Lecture #1
Tues, Sept 18, 2007

Title: "Energy & Electric Field"

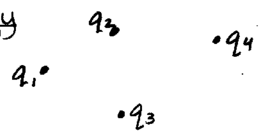
Recap: $\vec{F} = \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$ Force is a vector.

Now we define 2 properties of a system of charges:

U = electrostatic energy (scalar)

$\vec{E}$ = electric field (vector)

<u>Energy</u>



What's the energy of this system of charges?

First we note that the energy is unique, depending just on the configuration of charges (and not on how they got there in the first place).

This is because the electrostatic force is <u>conservative</u>.

Recall another conservative force: gravity

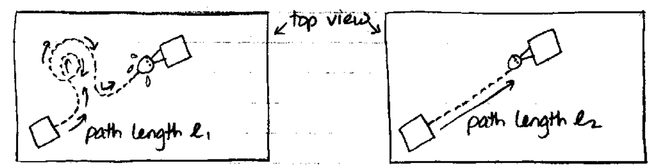
$W = \int$ Force $\cdot$ distance

$$W_1 = (mg\sin\theta_1)\sqrt{L_1^2+h^2} = mg\frac{h}{\sqrt{L_1^2+h^2}}\sqrt{L_1^2+h^2} = mgh$$

$$W_2 = (mg\sin\theta_2)\sqrt{L_2^2+h^2} = mg\frac{h}{\sqrt{L_2^2+h^2}}\sqrt{L_2^2+h^2} = mgh$$

Even though Dude #2 had to use a larger force on his block, the total work performed by Dude #1 and Dude #2 was the same, and the final energy is the same, $U = mgh$.

Recall a non-conservative force: friction

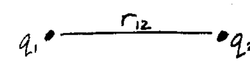


$W_1 = \mu_k mg\,\ell_1$ $\qquad$ $W_2 = \mu_k mg\,\ell_2$

$W_1 \gg W_2$ because $\ell_1 \gg \ell_2$
$\Rightarrow$ energy depends on path

Electric forces = conservative $\Rightarrow$ work doesn't depend on path

So let's look again at a system of charges:



What is the (unique) energy U?

We still need to define a reference energy, e.g. $U=0$ where $h=0$ at the bottom of the ramp in the box example.

For electric charges, we need to define <u>one</u> charge configuration where $U=0$, then we will talk about the energy of <u>every other</u> charge configuration with respect to that one.

We choose $U=0$ when all charges are infinitely far apart.

(Similar to our choice for gravity in planetary systems)

So, to compute the energy of

$q_1 \bullet \overset{r_{12}}{\text{———————}} \bullet q_2$

we need to compute the work required to bring q_2 in from ∞ to a distance r_{12}

$$W = \int_{r=\infty}^{r_{12}} \frac{-q_1 q_2}{r^2}\,\underbrace{dr}_{} = -q_1 q_2 \int_{\infty}^{r_{12}} \frac{dr}{r^2} = -q_1 q_2 \left[-\frac{1}{r}\right]\Bigg|_{\infty}^{r_{12}}$$

distance interval: always positive (sign is in the integration limits)

force you exert: equal and opposite the Coulomb force $= \frac{q_1 q_2}{r^2}$

This is the work you (an external agent) are doing.

$$= q_1 q_2 \left[\frac{1}{r}\right]\Bigg|_{\infty}^{r_{12}} = q_1 q_2 \left[\frac{1}{r_{12}} - \frac{1}{\infty}\right] = \frac{q_1 q_2}{r_{12}}$$

Do the sign math carefully! * eqn 3 in Purcell has a poor (wrong?) explanation of signs. Ignore!

If you have to do positive work (i.e. if q_1 & q_2 are like charges, so it's hard to bring them together), then the system increased in energy w.r.t. $U=0$ at $r=\infty$

$\Rightarrow U = W > 0$ (and you would get kinetic energy back by "dropping" the charges and letting them fly back to $U=0$ at $r=\infty$)

Finish with a physical common sense check on the signs.

[Note: if q_1 & q_2 are opposite signs, then you do negative work to bring them together, and $U<0$.]

Now let's add another charge q_3:

$q_1 \bullet \overset{r_{12}}{\text{———————}} \bullet q_2$

$\bullet \leftarrow$ label this point P_3

q_3

Adding q_3 at point P_3 takes work:

$$W_3 = \int_{\infty}^{P_3} -F_3 \cdot ds = -\int_{\infty}^{P_3} F_3 \cdot ds = -\int_{\infty}^{P_3} (F_{31} + F_{32}) \cdot ds$$

force done by you! = negative of Coulomb force

force on 3 from 1

force on 3 from 2

We get from here to here b/c addition of force vectors is linear!

$$W_3 = -\int_{\infty}^{P_3} F_{31} \cdot ds - \int_{\infty}^{P_3} F_{32} \cdot ds$$

but we already know how to compute these terms from our examination of 2 charges alone!

$\Rightarrow W_3 = \frac{q_1 q_3}{r_{31}} + \frac{q_2 q_3}{r_{32}}$ = work to bring in charge q_3 when charges q_1 & q_2 are already present & fixed

$\Rightarrow W_{tot} = \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{31}} = U$ = energy of whole 3-charge configuration

Note that this is symmetric in q_1, q_2, q_3

i.e. it didn't matter what order we put the charges in (we could have started with q_2 & q_3 present then pulled q_1 in last)

This is evidence for our assertion that the Coulomb force is a conservative force: the energy of the system depends only on the final configuration, not how we got there.

(This is not a full proof; for the full proof, see Purcell p12)

Generalize to N charges:

$$U = \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} \frac{q_j q_k}{r_{jk}}$$

don't worry about the self-energy of a single charge

factor of ½ because we double-counted terms like $\frac{q_j q_k}{r_{jk}}$ and $\frac{q_k q_j}{r_{jk}}$ in the sum

Electric Field

Now suppose you are a lonely charge q_0, walking into a bar full of charges.

What force will you feel?

(Let's suppose all the other charges are fixed i.e. they're glued in their seats and don't move in response to the force they feel from you. This is an important point - if they were moving, it would be a hard problem to solve self-consistantly!)

OK, you're going to feel a force from each of the N charges already sitting in the bar.

Luckily, these forces add linearly, so we can write:

$$\underbrace{\vec{F}_0}_{\text{vector force on you!}} \underbrace{(x,y,z)}_{\text{as a function of your position in the bar}} = \sum_{j=1}^{N} \frac{q_0 q_j}{r_{0j}^2} \hat{r}_{0j}$$

$\hat{r}_{0j}$: gives a vector direction to the force from each charge already in the bar (points from charge j towards you!)

Notice that q_0 is a factor in each force, so divide it out:

$$\vec{F}_0(x,y,z) = q_0 \underbrace{\sum_{j=1}^{N} \frac{q_j}{r_{0j}^2} \hat{r}_{0j}}_{\text{this depends only on the structure of the original charge system, not on you the lonely test charge}}$$

We define this quantity to be the electric field:

$$\vec{E}(x,y,z) = \sum_{j=1}^{N} \frac{q_j}{r_{0j}^2} \hat{r}_{0j}$$

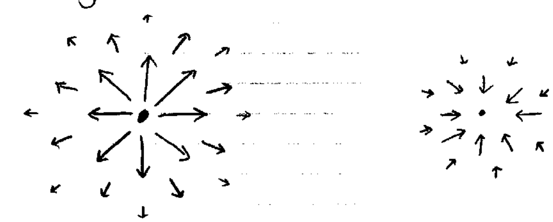
which has dimensions force/charge
and units dynes/esu

Electric field is a vector function of 3-dimensional space. It is a vector field.

For every point in space, $\vec{E}(x,y,z)$ tells you what any charge q will feel when it gets there, both a magnitude and a direction of force.

It's hard to draw this kind of thing!

Let's try to do it in 2 dimensions:

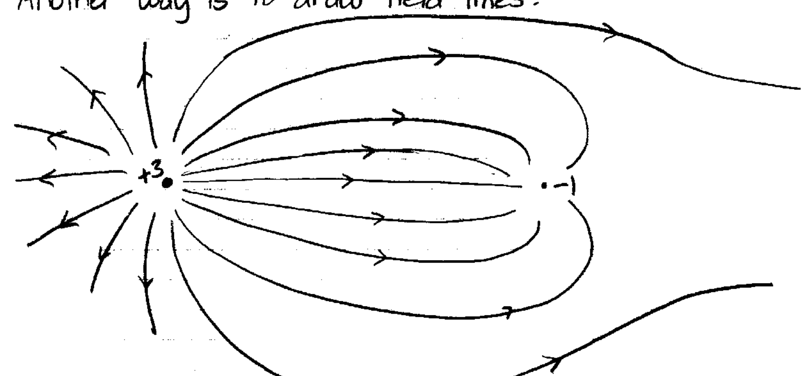


charge $q_1 = +3$ field points away | charge $q_2 = -1$ field points towards

For a representative set of points, we draw an arrow at each point where the length tells us the magnitude of the field and the direction tells us which way a positive test charge would move if we happened to drop it right there.

Another way is to draw field lines:

Here the density of field lines indicates the magnitude of field



Generalizing from discrete charge distributions to continuous charge distributions:

Electrons & protons are effectively point charges, but most things you deal with in daily life are macroscopic collections of so many electrons & protons that it doesn't make sense (nor is it computationally feasible) to continue to think in terms of point charges.

Instead, we talk about <u>charge density</u> ρ (rho)

$\rho(x,y,z)$ is a scalar function of position

$dq = \rho(x,y,z)\, dx\, dy\, dz$

is the small increment of charge contained in the small increment of volume $dx\,dy\,dz$

So we can write the electric field arising from a continuous charge distribution as:

$$\vec{E}(x,y,z) = \int \frac{\rho(x,y,z)\, dx\,dy\,dz}{r^2}\, \hat{r}$$

Next class: Gauss' Law

read Purcell chapter 1.9 - 1.15