Physics 15b (Hoffman)
Lecture #7
Tues, Oct 9, 2007

Title: "Capacitors"

Recap

<u>Conductors</u> have high conductivity σ (low resistivity $\rho = 1/\sigma$)
→ charge redistributes near-instantaneously

<u>Insulators</u> have low conductivity σ (high resistivity $\rho = 1/\sigma$)
→ charge stays put

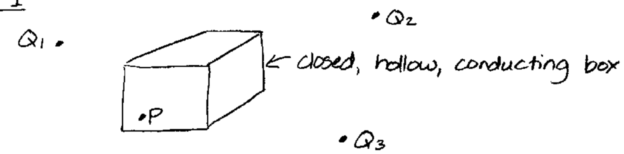
<u>5 properties of conductors</u>:

① $\vec{E} = 0$ in the interior of a conducting material
② $\vec{E}_{\parallel} = 0$ parallel to the surface of a conductor
③ φ is constant throughout a conductor (including the surface)
④ $\rho = 0$ within the conducting material
volume charge density
⑤ $\vec{E}_{\perp} = 4\pi\sigma$ *just* outside the conducting material
surface charge density

<u>Laplace's equation</u>: $\nabla^2 \varphi = 0$

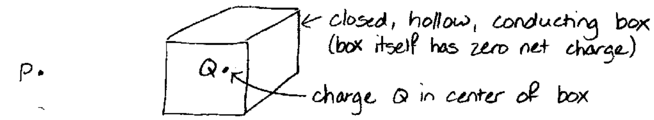
→ describes the electrostatic potential in a charge-free region
→ smoothly "averages" between boundary conditions
→ $\varphi(P)$ at the center of a sphere
= average of φ over the surface of any sphere centered at P
→ can solve numerically by iteratively averaging neighboring points on a grid
→ the solution is <u>unique</u> with appropriate <u>boundary conditions</u>

<u>Exercise 1</u>



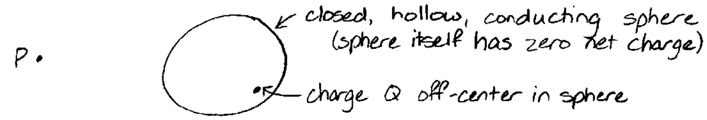
What is $\vec{E}$ at point P, inside the box?

<u>Exercise 2</u>



What is $\vec{E}$ at point P, outside, a distance R from box center?

<u>Exercise 3</u>



What is $\vec{E}$ at point P, outside, a distance R from sphere center?

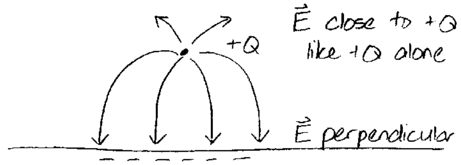
Goals for today

① Method of images: a point charge near a conducting plane
② Define capacitance
③ Energy stored in a capacitor

Image charges: a cute mathematical trick

Example: bring a point charge $+Q$ near a conducting plane; how does the charge redistribute in the plane, and what will be the resultant $\vec{E}$-field?

Qualitatively:

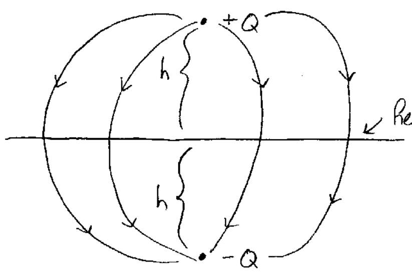


$\vec{E}$ close to $+Q$ looks just like $+Q$ alone (radial)

$\vec{E}$ perpendicular to surface

build up negative charge

But how to solve quantitatively?
Answer: apply a trick, method of images
(This method is generally useful, not some one-time brain real estate waster.)



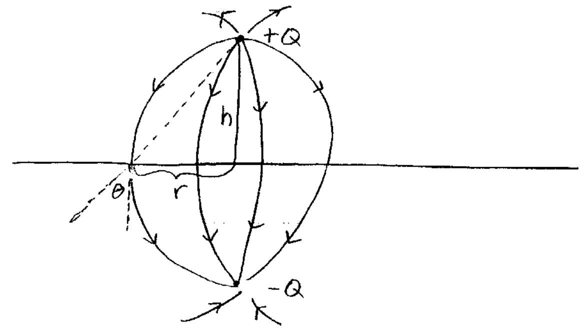
field lines are everywhere perpendicular to the middle plane

In the upper half plane, the field meets all the requirements of the actual problem:

$\vec{E}$ is perpendicular to plane of conductor
looks like point charge near $+Q$

By the uniqueness thm, if this is one solution in the upper half space, it must be the only solution.

(Note: slightly different boundary conditions than the uniqueness thm we proved: here we know $\varphi = 0$ on plane and at infinity, but instead of knowing φ by the point charge, we know the value of the charge. There's an analogous uniqueness thm for this situation → maybe prove for homework?)



$\vec{E}$ just above plane:

$$E_z = \frac{-2Q}{r^2+h^2}\cos\theta = \frac{-2Q}{r^2+h^2}\cdot\frac{h}{(r^2+h^2)^{1/2}} = \frac{-2Qh}{(r^2+h^2)^{3/2}}$$

This tells us the surface charge density:

$$\sigma = \frac{E_z}{4\pi} = \frac{-Qh}{2\pi(r^2+h^2)^{3/2}}$$

What is the total surface charge?

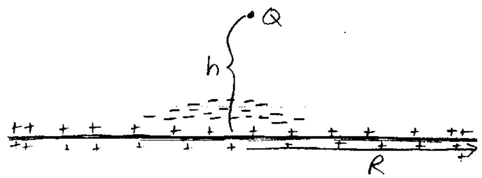
$$Q_{surf} = \int_0^\infty \sigma\, 2\pi r\, dr = -Q\int_0^\infty \frac{hr}{(r^2+h^2)^{3/2}}\,dr = -Q$$

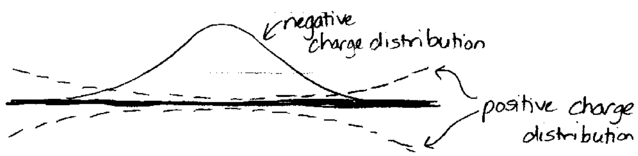
hint: if you're too lazy or can't remember how to do an integral, go to

http://www.integrals.com

But what if our conductor started uncharged?
How can it now have charge $-Q$?

There must be an equal and opposite $+Q$ distributed in plane





What is the force between the point charge and the conducting plane?

$\vec{F} = Q\vec{E}$, and $\vec{E}$ is as if there were a charge $-Q$ located a distance $2h$ away

$$\Rightarrow F = Q\left[\frac{-Q}{(2h)^2}\right] = \frac{-Q^2}{4h^2}$$

How much energy would it take to remove this point charge to infinity?

Method 1 To separate 2 point charges to infinity from a distance $2h$ would take work

$$W = Q\varphi$$

potential due to the other point charge at a distance $2h$

$$= Q\frac{Q}{2h} = \frac{1}{2}\frac{Q^2}{h}$$

Method 2

$$W = -\int F\,dx = -\int_h^\infty \frac{-Q^2}{4x^2}\,dx = \frac{Q^2}{4}\left(\frac{-1}{x}\right)\Big|_h^\infty = \frac{Q^2}{4h}$$

this negative sign is b/c we're computing the work that you, the external agent do to oppose the Coulomb Force F

Force and distance of motion point in opposite directions so this dot product is negative

Why do these methods not agree?

In the first method, $Q^2/2h$ would be the total work done to separate 2 charges. But one of these charges is fictitious. To remove the single real charge requires only half of this work.

Capacitance

Consider an isolated conductor carrying charge Q.
If we take $\varphi = 0$ at infinity, then the potential φ_0 is fixed on the conductor.

Now imagine we double Q, what happens to the potential?

Answer: it doubles too.
Why? The second Q must distribute itself on the conductor in the same way as the first Q, so now the charge density at each point on the surface is exactly twice what it was before. So the $\vec{E}$-field outside the conductor is twice what it was before. So the work (per charge) required to bring a test charge in from infinity (which is the definition of potential) must also double.

So, we've convinced ourselves that $\varphi \propto Q$
(potential is proportional to charge)

We could just as well state this the other way: $Q \propto \varphi$

In fact, let's name this constant: $Q = C\varphi$

"capacitance" of a given conductor

Exercise: What is the capacitance of the earth?
Assume the Earth is a reasonably good conductor (it is).
Radius of the Earth: $R = 6{,}378$ kilometers

cgs: If we put charge Q on the earth, the earth will have potential

$$\varphi = \frac{Q}{R} \quad \text{(with } \varphi = 0 \text{ at infinity)}$$

Compare this to $Q = C\varphi$

We see that $C = R = 6{,}378$ km
What!?! units of distance?!?
Yup, that's the cgs unit of capacitance
It makes some sense when you realize that capacitance of a conductor just scales with the size of that conductor. If you double the size of a conductor, it can hold twice as much charge at the same potential.

SI: Now if we put charge Q on the earth, the earth will have potential

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} \quad \text{(with } \varphi = 0 \text{ at infinity)}$$

$$\varepsilon_0 = 8.85\times10^{-12}\ C^2/N\cdot m^2$$

Compare this to $Q = C\varphi$, and we see that now

$$C = 4\pi\varepsilon_0 R = 4\pi(8.85\times10^{-12}\ C^2/N\cdot m^2)(6.378\times10^{6}\ m)$$

$$= 0.00071\ \underbrace{C^2/N\cdot m}_{\text{"Farad" = unit of capacitance in SI}}$$

Apparently 1 Farad is a ridiculously large unit if a sphere the size of the earth has $C = 0.7$ mF.
In fact, usually we deal in pF (picoFarad = 10^{-12} Farad) and occasionally one sees in the lab a capacitor as large as 1 μF (microFarad = 10^{-6} Farad).

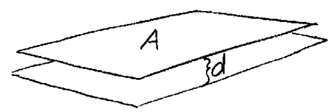
Parallel plate capacitors

Now in the real world, charge is conserved, so you can't just give one conductor charge Q without taking it away from some place else. So usually we deal with "capacitors" which are objects with 2 conductors carrying equal and opposite charge $\pm Q$.

For a given $\pm Q$, there is a given potential difference between these charged conductors, which will be proportional to Q. So now we have:

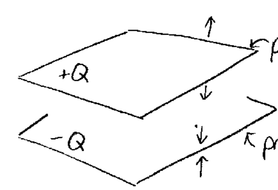
$$Q = \underset{\uparrow \text{capacitance}}{C}(\varphi_1 - \varphi_2) = C\Delta\varphi$$

What is the C of a parallel plate capacitor of area A and separation distance d?



cgs: Suppose we put $+Q$ on the top plate and $-Q$ on the bottom plate. Then the charge density is $\sigma = \pm\frac{Q}{A}$ on the two plates.

We know from a previous result (using Gauss' Law) that the field by an infinite plane with surface charge σ is: $E = 2\pi\sigma$ pointing perpendicular to the plane on both sides



produces field $E = 2\pi\frac{Q}{A}$ down between the plates

produces field $E = 2\pi\frac{Q}{A}$ down between the plates

$\Rightarrow$ the total field between the plates is $E_{tot} = 4\pi\frac{Q}{A}$

(and the total field outside the plates vanishes)

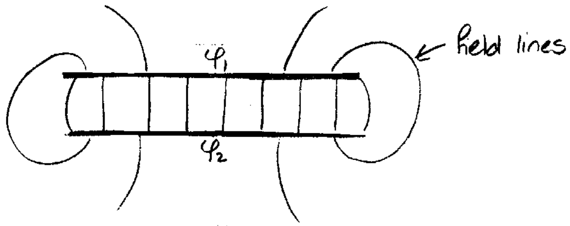
The potential difference between the plates is the work per charge required to bring a test charge from one plate to the other, through this field $E = 4\pi \frac{Q}{A}$

$$\Delta\varphi = \int_{\text{plate 1}}^{\text{plate 2}} E \cdot dx = \int_0^d 4\pi \frac{Q}{A} dx = 4\pi \frac{Q}{A} d$$

Compare this to $Q = C\,\Delta\varphi$

$$\Rightarrow C = \frac{A}{4\pi d}$$

Note: we assumed the plates were infinite. In reality they are finite and we will have edge effects:



But as long as $\sqrt{A} \gg d$, the edge effects are negligible.

Energy stored in a capacitor

Clearly we have done some work to charge a capacitor to $\pm Q$. How much?

Consider a capacitor C with charge $\pm Q$.
What is the incremental work to bring an additional dQ from the negative plate to the positive plate, so that the capacitor will now have charge $\pm(Q+dQ)$?

If we take dQ infinitesimal, so that the field it experiences as it moves between the plates is still the same as when the plates had charge $\pm Q$, then the work is just $dQ\,\Delta\varphi$ (this is the definition of potential).

So the work to charge the capacitor starting from scratch is:

$$W = \int_0^Q \Delta\varphi\, dQ'$$

but $\Delta\varphi = \frac{1}{C} Q'$ from the definition of capacitance

$$\Rightarrow W = \int_0^Q \frac{Q'}{C} dQ' = \frac{Q^2}{2C}$$

Exercise: verify this expression $U = \frac{1}{2}\frac{Q^2}{C}$ for the energy stored in a parallel-plate capacitor by integrating over the energy stored in the field.