Physics 15b (Hoffman)
Lecture # 22
Tues, Dec 11, 2007

Title: Magnetic dipoles

Goals for today:
- summarize & wrap up our discussion of $\vec{E}$-fields in matter
  - bound charge
  - bound current
  - speed of light in matter
- compare to materials in magnetic fields
- magnetic multipole expansion
  - magnetic monopoles?
  - force on a magnetic dipole
  - torque on a magnetic dipole
- microscopic response to applied B
  - diamagnetism
  - paramagnetism

Summary of last 3 lectures:

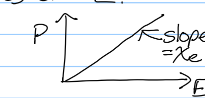
$\vec{p}$ = "dipole moment" = relative displacement of positive & negative charges

$$= \underbrace{\int \rho(\vec{r}\,')\vec{r}\,' d\tau}_{\text{volume of charge distribution}} = \sum_{\text{all point charges } q_j} q_j \vec{r}_j$$

$\vec{P}$ = "polarization" = dipole moment per unit volume
(e.g. $N\vec{p}$ where N = # of molecules/cm^3 and $\vec{p}$ = dipole moment of single molecule)

α = "polarizability" = tendency of a non-polar molecule to acquire a dipole moment in an applied $\vec{E}$-field, defined by $\vec{p} = \alpha\vec{E}$
typically $\alpha \sim a^3$ where a = dimension of molecule

χ_e = "dielectric susceptibility" = tendency of a macroscopic material to acquire a macroscopic polarization $\vec{P}$ in the presence of an applied $\vec{E}$
defined by $\vec{P} = \chi_e \vec{E}$
Note: not all materials obey this nice linear relation, but most do, for reasonable ranges of $\vec{E}$.
For a linear material: $\chi_e = \frac{\varepsilon - 1}{4\pi}$



ε = "dielectric constant" - property of a material such that:
1. if material is stuck between capacitor plates, the charge on the plates is increased from vacuum value Q_0 to $Q = \varepsilon Q_0$
2. more generally, electric fields produced by free charges in the material are reduced by a factor ε with respect to vacuum

$\vec{D}$ = "displacement current" $\equiv \vec{E} + 4\pi\vec{P}$
For a linear material, this reduces to
$$\vec{E} + 4\pi\left(\frac{\varepsilon - 1}{4\pi}\right)\vec{E} = \varepsilon\vec{E}$$
$\vec{D}$ is a convenient definition that allows us to simplify Maxwell's laws in matter.

Bound charge

Let's make a distinction between the "free charge" Q that we intentionally placed (that came out of our battery, etc) and the "bound charge" which comes from the dielectric response.

We could rewrite Gauss' law for our point charge above as:

$$\oint \varepsilon \vec{E} \cdot d\vec{a} = 4\pi Q_{free}$$

By superposition of point charges, this holds for any free charge distribution:

$$\oint_S \varepsilon \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho_{free}\, d\tau \Rightarrow \vec{\nabla} \cdot (\varepsilon \vec{E}) = 4\pi \rho_{free} \qquad (1)$$

But Gauss' law still holds: $\vec{\nabla} \cdot \vec{E} = 4\pi \rho_{tot} = 4\pi(\rho_{free} + \rho_{bound})$ (2)

Subtract (2) from (1): $\vec{\nabla} \cdot (\varepsilon - 1)\vec{E} = -4\pi \rho_{bound}$

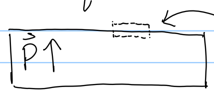
but from $\chi_e = \frac{P}{E} = \frac{\varepsilon - 1}{4\pi}$, we have: $\vec{\nabla} \cdot \vec{P} = -\rho_{bound}$ (3)

$$\Rightarrow \vec{\nabla} \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho_{free} \qquad (4)$$

At first look, (4) does not seem like progress from (1). But it turns out that (4) is always valid, even for nonlinear materials where P is not proportional to E.

We define $\vec{D} \equiv \vec{E} + 4\pi \vec{P}$ = "displacement vector".
In uniform, linear dielectrics, $\vec{D} = \varepsilon \vec{E}$
But $\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{free}$ is more generally useful.

Example What is surface charge on a uniformly polarized dielectric slab of polarization $\vec{P}$?
Answer: use eqn (3) above



From this infinitesimal box at the surface, we have $\vec{P} = 0$ on top surface, $\vec{P}$ pointing out on bottom surface, and no $\vec{P}$ pointing in or out on sides, so $\vec{\nabla} \cdot \vec{P} = -\rho$

$\Rightarrow \rho_{bound} = P \Rightarrow \sigma_{bound}$ on top $= P$ (as we already knew)

Bound charge current

Suppose we have N dipoles per cm^3 and that in time dt each changes from $\vec{p}$ to $\vec{p} + d\vec{p}$ as each q moves by $d\vec{s}$ in time dt, i.e. $d\vec{p} = q\, d\vec{s}$.

$\Rightarrow \vec{P}$ changes from $N\vec{p}$ to $N\vec{p} + N d\vec{p}$

and there is also a real current:

$$\vec{J} = Nq \frac{d\vec{s}}{dt} = N \frac{d\vec{p}}{dt} = \frac{d\vec{P}}{dt}$$

This is model-independent: changing $\vec{P}$ always causes $\vec{J} = \frac{d\vec{P}}{dt}$

Any $\vec{J}$ also causes $\vec{B}$, so we re-examine Maxwell's eqn:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

We can now think of $\vec{J}$ in 2 parts: the "free current" which we control externally, and the "bound current" which results from changing polarization in a material.

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}_{free} + \underbrace{\frac{4\pi}{c}\frac{\partial \vec{P}}{\partial t}}_{\text{bound charge current density term}}$$

But we had previously defined $\vec{D} \equiv \vec{E} + 4\pi \vec{P}$, and noted that for a linear dielectric, $\vec{D} = \varepsilon \vec{E}$.

$$\Rightarrow \underbrace{\vec{\nabla} \times \vec{B} = \frac{1}{c}\left(\frac{\partial \vec{D}}{\partial t} + 4\pi \vec{J}_{free}\right)}_{\text{always}} \underbrace{= \frac{1}{c}\left(\varepsilon \frac{d\vec{E}}{dt} + 4\pi J_{free}\right)}_{\text{for uniform, linear materials}}$$

EM waves in dielectric

Start from Maxwell's equations:

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}_{free}$$

To simplify matters, we're not going to worry about boundary conditions for now; just assume we have an infinite expanse of uniform, linear dielectric material. In this case, we have $\vec{D} = \varepsilon\vec{E}$, and $\rho_{free} = 0$ and $\vec{J}_{free} = 0$

So Maxwell's equations reduce to:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t}$$

We can procede the same way to eliminate $\vec{B}$:

$$\underbrace{\vec{\nabla}\times\vec{\nabla}\times\vec{E}}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}} = -\frac{\varepsilon}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \nabla^2\vec{E} = \frac{\varepsilon}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Likewise, $\nabla^2\vec{B} = \frac{\varepsilon}{c^2}\frac{\partial\vec{B}}{\partial t}$

These equations still have solutions of the form

$$\vec{E} = \hat{z}E_0\sin(ky-\omega t)$$
$$\vec{B} = \hat{x}B_0\sin(ky-\omega t)$$

But now we find that $k^2 = \frac{\varepsilon}{c^2}\omega^2$

$\Rightarrow v = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon}}$ so the speed of light is reduced by $1/\sqrt{\varepsilon}$ in a dielectric

Compare how materials react to electric vs. magnetic fields:

$\vec{E}$-field

① material always polarizes the same direction in applied $\vec{E}$
maybe not linear, but we do know that $\vec{P}$ is always going to be parallel to $\vec{E}$

microscopically:
non-polar molecule: will always charge-separate in the direction of field:

plus charge must go in direction of $\vec{E}$; that's the definition of $\vec{E}$!
$\vec{E}$ is defined such that $\vec{F} = q\vec{E}$

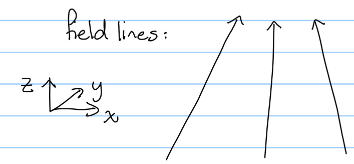
polar molecule: will always have lowest energy configuration aligned with the field

$\vec{\tau} = \vec{p}\times\vec{E} = pE\sin\theta = 0$ only when $\theta = 0$
(i.e. until $\theta = 0$ there will be a torque telling θ to go to zero)

② $\vec{F}_{net} = 0$ for a dipole in a uniform $\vec{E}$-field

③ $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$ for a dipole in a non-uniform $\vec{E}$-field

field lines:



Here, $\vec{\nabla}E_z$ has a gradient in the $+\hat{z}$ direction: the field lines are denser for larger z.

If we let $\vec{p} = p\hat{z}$, it will be attracted towards stronger field.
If we let $\vec{p} = -p\hat{z}$, it will be repelled from stronger field
But if $\vec{p}$ is polarized by $\vec{E}$ to begin with (i.e. if $\vec{p} = 0$ before $\vec{E}$ came along) then $\vec{p}$ must lie in the direction of $\vec{E}$, so $\vec{p}$ must be attracted to stronger $\vec{E}$-field.
$\Rightarrow$ macroscopically polarized materials* will always be attracted towards regions of stronger fields

* assuming they are polarized by $\vec{E}$ to begin with and not spontaneously polarized in a random direction

B-field:

		"relative permeability"
zinc chloride	$ZnCl_2$	−65
bismuth	Bi	−280
aluminum sulfate	$Al_2(SO_4)_3 \cdot 18H_2O$	−323
copper sulfate	$CuSO_4$	+1330
manganese sulfate	$MnSO_4 \cdot H_2O$	+14200
iron	Fe	ferromagnetic
liquid oxygen	O_2	+HUGE

Demo: some materials are attracted, some are repelled from regions of stronger field.
For most materials the orientation of the material itself will not determine whether it is attracted or repelled.
Iron is an exception.

⇒ we seem to have 3 categories of response of material to a magnetic field... not as simple as $\vec{E}$

diamagnetism: repelled by stronger field

paramagnetism: attracted to stronger field

ferromagnetism: depends on the orientation of the material itself

Let's understand the microscopics first, and build up to the macroscopics, as we did for the $\vec{E}$-fields.

For $\vec{E}$-fields, we started by looking at the far field of a small charge distribution (e.g. molecule with lopsided e^- distribution)

What is the analog for $\vec{B}$?

Compare Maxwell's eqns: $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ vs. $\vec{\nabla}\cdot\vec{B} = 0$

charge = monopole in our expansion

? NO MAGNETIC MONOPOLES!

Magnetic Monopoles

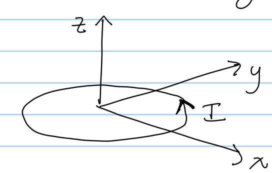
There is no such thing as a magnetic monopole (at least it has never been credibly observed.)

If you break an electric dipole in half, you actually get positive charge in one hand and negative in the other.

But if you break a magnet apart, you don't get north in one hand and south in the other; you just get 2 smaller magnets.

⇒ Simplest microscopic source of $\vec{B}$-field is a current loop

$\vec{B}$-field produced by a current loop



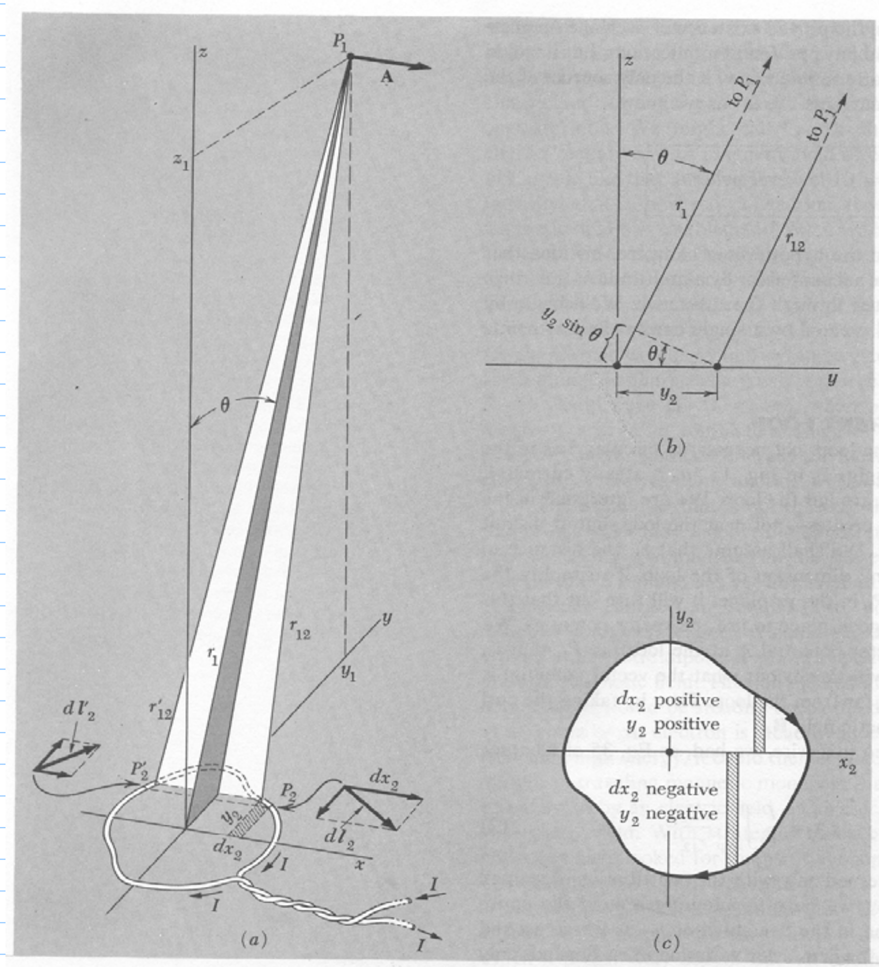
We computed this exactly for a symmetric current loop, way back in chapter 6, using Biot-Savart: $d\vec{B} = \frac{I\,d\vec{l}\times\hat{r}}{cr^2}$

But let's do this more generally for any current loop.

The best way to proceed in general is with the magnetic vector potential. Recall, for $\vec{E}$ we could define a scalar potential φ such that $\vec{E} = -\vec{\nabla}\varphi$. We could do this because for static fields, $\vec{\nabla}\times\vec{E} = 0$, so $\vec{E}$ is conservative. For $\vec{B}$ we have to resort to the vector potential $\vec{A}$, where $\vec{B} = \vec{\nabla}\times\vec{A}$.

We found that in general, $\vec{A}$ can be written in terms of $\vec{J}$:

$$\vec{A} = \underbrace{\frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{r}\,dv'}_{\text{bulk}} = \underbrace{\frac{I}{c}\int \frac{d\vec{l}}{r}}_{\text{current in a wire}}$$



We want to compute $\vec{A}$ at point P_1, far away, for the irregular current loop shown. Pick axes such that the current loop lies in the xy plane, and P_1 lies in the yz plane, so that:

$$\vec{A}(0, y_1, z_1) = \frac{I}{c}\int_{\text{loop}} \frac{d\vec{l}_2}{r_{12}}$$

First, let's simplify r_{12}:

$$r_{12} = \sqrt{x_2^2 + (y_1 - y_2)^2 + z_1^2}$$

$$= \sqrt{\underbrace{x_2^2 + y_2^2}_{r_2^2 = \text{small}} - 2y_1y_2 + \underbrace{y_1^2 + z_1^2}_{r_1^2 = \text{big}}}$$

$$= r_1\sqrt{\frac{r_2^2}{r_1^2} - \frac{2y_1y_2}{\underbrace{r_1^2}} + 1}$$

note that $y_1/r_1 = \sin\theta$

$$= r_1\sqrt{1 - \underbrace{\frac{2y_2}{r_1}\sin\theta}_{\text{this is small}} + \underbrace{\frac{r_2^2}{r_1^2}}_{\text{this is (small)}^2\text{, so we drop it}}}$$

$$\approx r_1\left(1 - \frac{2y_2}{r_1}\sin\theta\right)^{1/2}$$

Use $(1+\delta)^{1/2} \approx 1 + \frac{1}{2}\delta + \ldots$

$$r_{12} \approx r_1\left(1 - \frac{1}{2}\cdot\frac{2y_2}{r_1}\sin\theta\right) = r_1 - y_2\sin\theta$$

$$\frac{1}{r_{12}} \approx \frac{1}{r_1 - y_2\sin\theta} = \frac{1}{r_1}\left(\frac{1}{1 - y_2/r_1\,\sin\theta}\right) = \frac{1}{r_1}\left(1 - \frac{y_2}{r_1}\sin\theta\right)^{-1}$$

Use $(1+\delta)^{-1} \approx 1 - \delta + \ldots$

$$\frac{1}{r_{12}} \approx \frac{1}{r_1}\left(1 + \frac{y_2}{r_1}\sin\theta\right)$$

Ok, so we have:

$$\vec{A}(0, y_1, z_1) = \frac{I}{c}\int_{\text{loop}} \frac{d\vec{l}_2}{r_1 - y_2\sin\theta} = \frac{I}{c}\int_{\text{loop}} \frac{d\vec{l}_2}{r_1}\left(1 + \frac{y_2}{r_1}\sin\theta\right)$$

Now let's simplify the direction of the vector:

Look at points P_2 and P_2', at the same y_2 but on opposite sides of the y-axis. For these points, the denominator $\frac{1}{r_{12}} \approx \frac{1}{r_1}\left(1 + \frac{y_2}{r_1}\sin\theta\right)$ will be the same, and the dy_2's (y-components of $d\vec{l}_2$) will be equal and opposite

We conclude that all of the dy_2 components will cancel, so we are left only with dx_2 components.

$$\vec{A}(0, y_1, z_1) = \hat{x}\frac{I}{c}\underbrace{\oint_{loop} \frac{1}{r_1}}_{\text{can pull } r_1\text{'s outside}}\left(1 + \frac{y_2}{r_1}\sin\theta\right)dx_2$$

can pull $\sin\theta$ outside too, because it just depends on r_1

$$= \hat{x}\frac{I}{cr_1}\underbrace{\oint_{loop} dx_2}_{\text{vanishes around loop}} + \hat{x}\frac{I}{cr_1^2}\sin\theta\underbrace{\oint_{loop} y_2\,dx_2}_{\text{area of loop}}$$

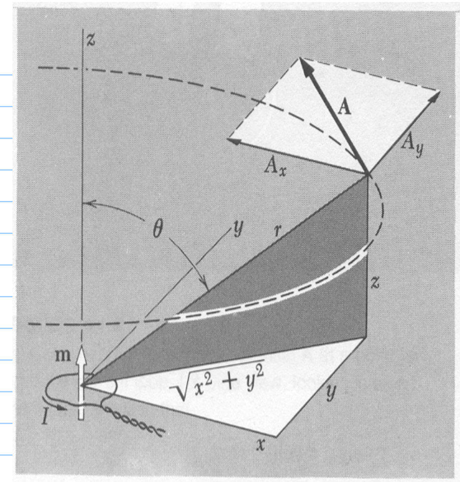
$$\Rightarrow \vec{A}(0, y_1, z_1) = \frac{Ia\sin\theta}{cr_1^2}\hat{x}$$

Note that the shape of the loop dropped out!

We define $\vec{m} = \frac{I\vec{a}}{c}$ = dipole moment of loop

$$\Rightarrow \vec{A} = \frac{\vec{m}\times\hat{r}}{r^2}$$

Now we need to compute $\vec{B}$ from $\vec{B} = \vec{\nabla}\times\vec{A}$



$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{3mxz}{r^5}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{3myz}{r^5}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{m(3z^2 - r^2)}{r^5}$$

Now we can compute the force on the dipole from a non-uniform field:

$$F = \frac{2\pi r I B_r}{c} \quad ; \quad 2\pi r(\Delta z)B_z = \text{net outward flux from sides}$$

$$\pi r^2\left[-B_z(z) + B_z(z+\Delta z)\right] = \text{net outward flux from ends}$$

$$= \pi r^2 \frac{\partial B_z}{\partial z}\Delta z \quad \Rightarrow \quad B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

$$\Rightarrow F = \frac{2\pi r I}{c}\frac{r}{2}\frac{\partial B_z}{\partial z} = \frac{\pi r^2 I}{c}\frac{\partial B_z}{\partial z} = m\frac{\partial B_z}{\partial z}$$