

Physics 15b (Hoffman)  
Lecture #20  
Tues, Dec 4, 2007

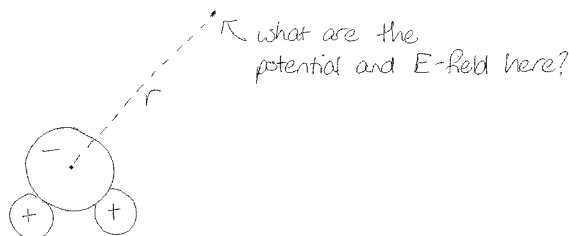
# Title: "Electric Polarization"

Goal: understanding electric fields in matter (part 2 of 3)  
Motivation: in order to understand how 300 Farads of capacitance could fit into that very small cylinder  $\rightarrow$  the matter inside the capacitor must be doing something very strange!

Plan of attack:

- understand single molecules
- put them together to understand macroscopic matter

Last time: we tried to understand the far-field behavior of a localized charge distribution.



We expanded the potential  $\phi(\vec{r})$  as a sum of multipole moments, falling off with inverse powers of  $r$ :

$$\phi(\vec{r}) = \underbrace{\frac{q_{tot}}{r}}_{\text{monopole}} + \underbrace{\frac{\hat{r} \cdot \vec{p}}{r^2}}_{\text{dipole}} + \underbrace{\frac{1}{r^3} K_2 + \dots}_{\text{quadrupole}}$$

Let's do one more example for how to compute  $\vec{p}$ .

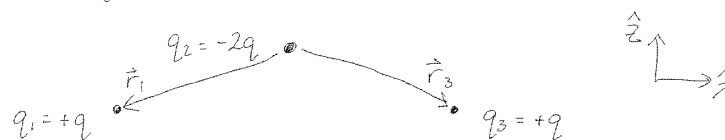


What is  $\vec{p}$ ?

No net charge, so we can put our origin anywhere.  
 $\rightarrow$  let's put our origin at the location of the  $-2q$  (it's often easiest to pick the most symmetric point as the origin)

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau' \rightarrow \sum_{\substack{\text{all discrete} \\ \text{charges } q_j}} q_j \vec{r}_j$$

continuous charge distrib.

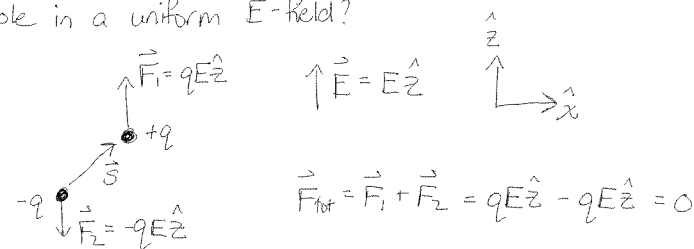


With our origin chosen at  $q_2$ ,  $\vec{r}_2 = 0$ , so  $q_2$  does not contribute to the dipole sum. Furthermore, the  $x$ -components of the  $q_1$  and  $q_3$  contributions cancel, so we are left with:

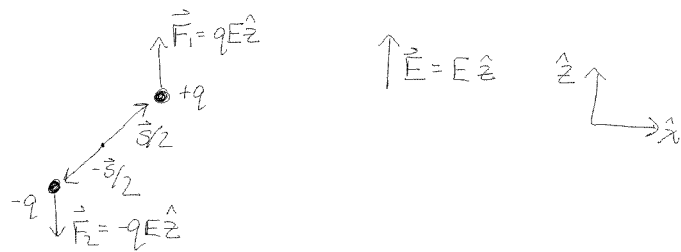
$$\vec{p} = 2q(-d \cos 60^\circ \hat{z}) = 2q(-\frac{1}{2}d \hat{z}) = -qd \hat{z}$$

How does a dipole behave in an applied field?

Force on a dipole in a uniform  $\vec{E}$ -field?

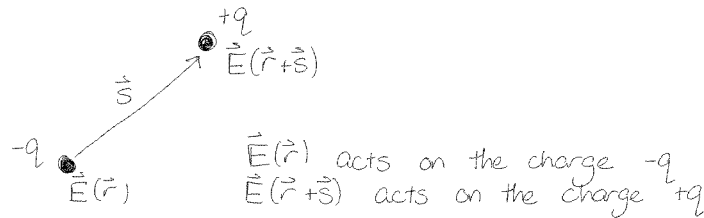


Torque on a dipole in a uniform  $\vec{E}$ -field?



$$\begin{aligned}\vec{N}_{\text{tot}} &= \sum \vec{r} \times \vec{F} = \frac{\vec{s}}{2} \times qE\hat{z} + \left(-\frac{\vec{s}}{2}\right) \times (-qE\hat{z}) \\ &= q\vec{s} \times \vec{E}\hat{z} = \vec{p} \times \vec{E}\end{aligned}$$

Force on a dipole in a non-uniform  $\vec{E}$ -field?



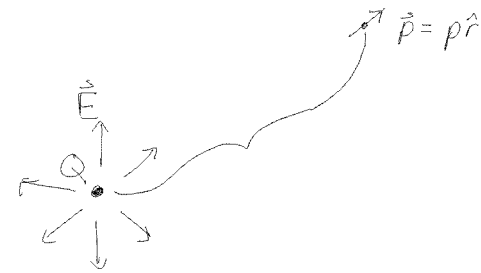
We assume that  $\vec{s}$  is small compared to the length scale over which  $\vec{E}$  changes significantly, so we expand  $\vec{E}(\vec{r}+\vec{s})$  for small  $\vec{s}$ :

$$\begin{aligned}\vec{F}_{\text{tot}} &= -q\vec{E}(\vec{r}) + q\vec{E}(\vec{r}+\vec{s}) \\ &= q\left[\vec{E}(\vec{r}) + s_x \frac{\partial \vec{E}}{\partial x} + s_y \frac{\partial \vec{E}}{\partial y} + s_z \frac{\partial \vec{E}}{\partial z}\right] \\ &= q\vec{E}(\vec{r}) + (q\vec{s} \cdot \vec{\nabla})\vec{E}\end{aligned}$$

$$\vec{F}_{\text{tot}} = -q\vec{E}(\vec{r}) + q\vec{E}(\vec{r}) + (\vec{p} \cdot \vec{\nabla})\vec{E}$$

$$\vec{F}_{\text{tot}} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

Example: what is the force on a dipole  $\vec{p}$  from a point charge  $Q$ , if  $\vec{p}$  is oriented radially?



$$\vec{F} = \left[ p\hat{r} \cdot \left( \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi} \right) \right] \vec{E}$$

gradient operator in spherical coords

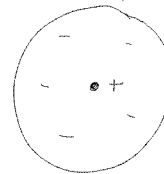
$$= p \frac{\partial}{\partial r} \vec{E} = p \frac{\partial}{\partial r} \left( \frac{Q}{r^2} \hat{r} \right) = -\frac{2pQ}{r^3} \hat{r}$$

The force points radially inward because the negative part of the dipole is closer to the point charge, so it feels the stronger force.

Types of dipoles:

① permanent dipoles, like water molecules  
→ these will rotate to align with applied  $\vec{E}$

② induced dipoles, e.g. hydrogen atom



positive nucleus & negative charge cloud  
→ in an applied field, the negative charge cloud will shift slightly so that its center is no longer aligned with the positive nucleus

induced dipole  $\rightarrow \vec{p} = \alpha \vec{E}$  ← applied field  
"polarizability"

Estimate the polarizability of a typical atom:

In response to an applied field  $\vec{E}$ , the "center of gravity" of the negative charge cloud will shift by  $\Delta z$ .

$\Delta z$  is limited by the restoring force of the positive nucleus pulling back on the negative charge cloud.

If  $a$  is a typical atom dimension, then  $E_{int} \sim e/a^2$  is a typical internal field holding the atom together.

The fractional distortion of the atom in the presence of the applied field should be equal to the fractional change in the total field experienced by the charge cloud in the presence of the applied external field:

$$\frac{\Delta z}{a} \sim \frac{E_{applied}}{E_{internal}} = \frac{E_{app}}{e/a^2}$$

$$\Rightarrow \Delta z \sim \frac{a^3}{e} E_{app}$$

$$\vec{p} = e \Delta z \sim a^3 \vec{E}_{app}$$

$$\Rightarrow \alpha \equiv \text{polarizability} \sim a^3$$

The main point to take from this is that for a typical atom (or non-polar molecule) the induced dipole moment is to a good approximation linear in the applied field. In other words the polarizability  $\alpha$  is pretty constant with applied field.

Ok, we've spent some time figuring out how a dipole will respond to its environment.

Now let's figure out what a dipole does to its environment. Furthermore, let's put a bunch of dipoles together to make a macroscopic piece of material and see how that behaves.

Capacitors, again...

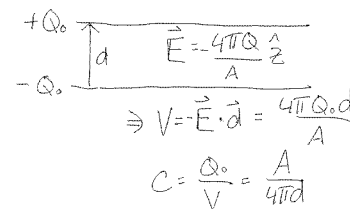
So we know we have this huge capacitor, with  $C$  much bigger than we can explain with our old geometric formula  $C = \frac{A}{4\pi d}$  (cgs) or  $\frac{\epsilon_0 A}{d}$  (SI)

What does this mean, that  $C$  is too big?

$C = Q/V$ , the amount of charge you can pile onto the plates for a given applied voltage

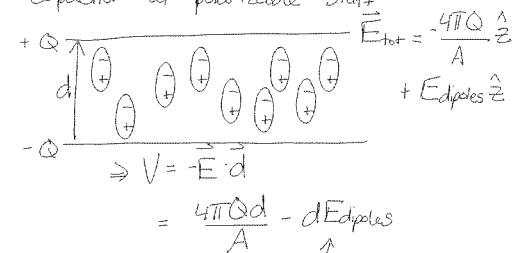
Now suppose the space between the plates is filled with a polarizable material (either nonpolar stuff that polarizes, or permanent dipoles that rotate). As we start to apply  $V$  to the capacitor, the plates start to charge, but the material inside starts to partially cancel the field from the charged plates, so more charge needs to flow onto the plates to keep the same  $V$ . This is why the ratio  $C = Q/V$  can be larger when there is a material inside.

vacuum capacitor



$$\begin{aligned} \vec{E} &= -\frac{4\pi Q_0}{A} \hat{z} \\ \Rightarrow V &= -\vec{E} \cdot \vec{d} = \frac{4\pi Q_0 \cdot d}{A} \\ C &= \frac{Q_0}{V} = \frac{A}{4\pi d} \end{aligned}$$

capacitor w/ polarizable "stuff"



$$\begin{aligned} \vec{E}_{tot} &= -\frac{4\pi Q}{A} \hat{z} + E_{dipoles} \hat{z} \\ \Rightarrow V &= -\vec{E} \cdot \vec{d} \\ &= \frac{4\pi Q d}{A} - d E_{dipoles} \end{aligned}$$

because of this negative term, we see that in order to arrive at the same voltage, we'll need a much larger  $Q$ ,  $\therefore$  a larger  $Q/V$  ratio

Let's quantify this: calculate the electric field caused by a macroscopic amount of polarized matter

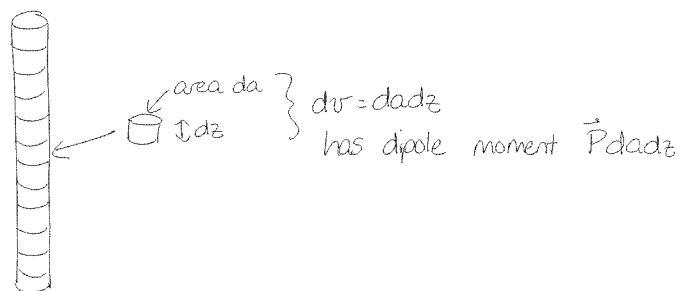
Suppose we have  $N$  dipoles per  $\text{cm}^3$ , each of strength  $\vec{p}$

Assume  $N$  is large, so any "macroscopically small" volume  $d\tau$  still contains many, many dipoles

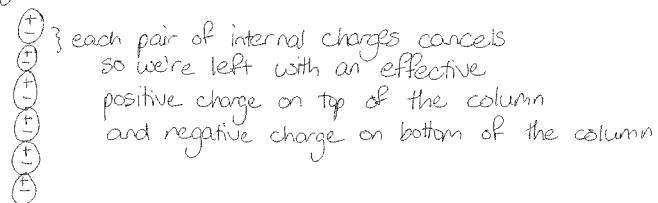
$\Rightarrow$  volume  $d\tau$  has total dipole moment  $N\vec{p}d\tau$

Define  $\vec{P} \equiv N\vec{p}$  = density of polarization

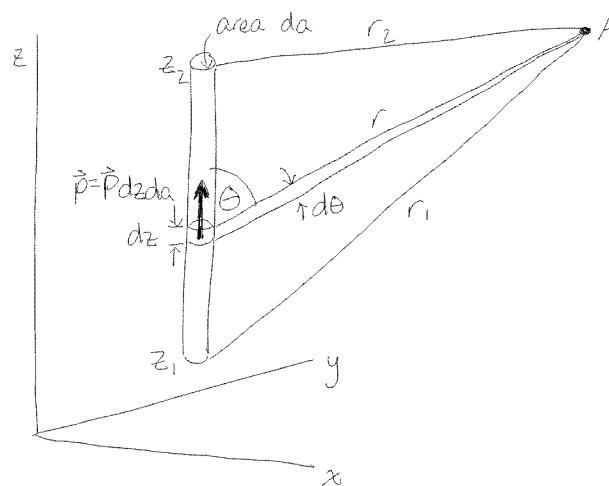
Consider a slender column of polarizable material:



so we can think of this as a stack of dipoles, each of strength  $\vec{P}da dz$



need to make this rigorous...

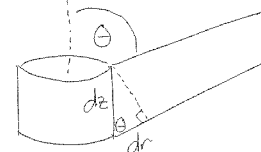


$$d\phi_A = \frac{\hat{r} \cdot \vec{p}}{r^2} = \frac{P da dz \cos \theta}{r^2}$$

note the negative sign:  
r gets smaller as  
z increases

$$\phi_A = \int_{z_1}^{z_2} \frac{P da dz \cos \theta}{r^2}$$

note:  $dz \cos \theta = -dr$



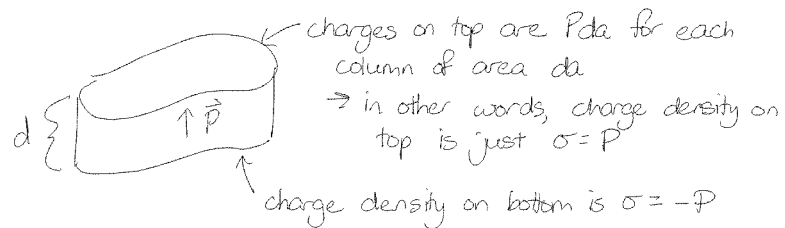
$$\Rightarrow \phi_A = \int_{r_1}^{r_2} P da \frac{-dr}{r^2} = P da \left( \frac{1}{r} \right) \Big|_{r_1}^{r_2} = P da \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

$$= \underbrace{P da}_{r_2} - \underbrace{P da}_{r_1}$$

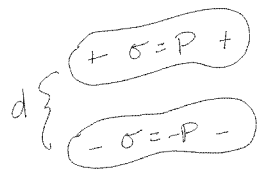
potential of  
a positive charge  
 $P da$  located at  
top of column

potential of  
a negative charge  
 $-P da$  located at  
bottom of column

We can automatically extend this result to a slab of polarized material (a bunch of columns in parallel)



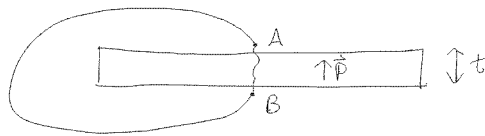
$\Rightarrow$  slab of polarized material is equivalent (from an outside perspective) to 2 charged plates



What about the field inside the slab?

microscopically, it's going to be a mess, dominated by the strong fields (on order 100s of kV/cm) from individual nuclei and electrons

Consider a slab, viewed from the side

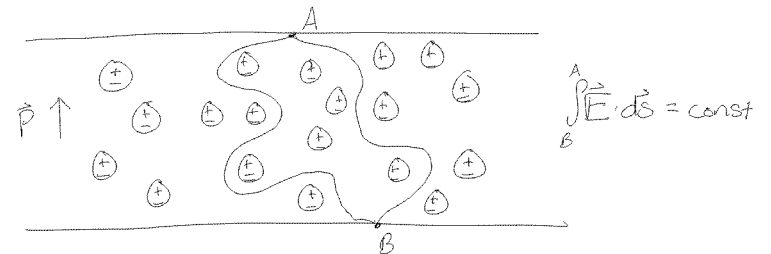


$\int_A^B \vec{E} \cdot d\vec{s}$  must be path-independent

$\rightarrow$  it's determined by the external field which is just that of 2 charged plates

$$\Rightarrow \int_A^B \vec{E} \cdot d\vec{s} = \varphi_B - \varphi_A = 4\pi\sigma t = 4\pi Pt$$

What we're saying is that no matter how we get from A to B, the total path integral must be the same



But this  $\vec{E}$  is the total field, including the messy microscopic fields, and the path integral of this messy field, no matter which way we go, still gives us the clean value  $4\pi Pt$ .

$\Rightarrow$  We have to conclude that the spatially averaged field within the polarized slab must be  $-4\pi P$

denote  $\langle \vec{E} \rangle_V$  = spatial average of  $\vec{E}$  over volume  $V$

$$\langle \vec{E} \rangle_V = \frac{1}{V} \int_V \vec{E} d\tau$$

$\uparrow$  "macroscopic field"       $\uparrow$  "microscopic field"

It turns out that  $\langle \vec{E} \rangle$  is really the only field we can or need to talk about inside a material

$\int_A^B \langle \vec{E} \rangle \cdot d\vec{s}$  is path-independent

$$\Rightarrow \vec{\nabla} \times \langle \vec{E} \rangle = 0$$

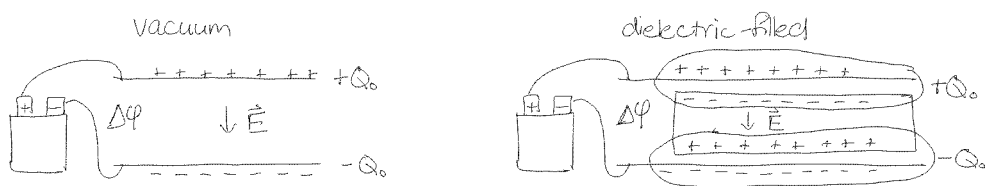
$$\text{and } \langle \vec{E} \rangle = -\vec{\nabla} \langle \varphi \rangle$$

for some potential function  $\langle \varphi \rangle$

We also state without proof that  $\nabla \cdot \langle \vec{E} \rangle = 4\pi \langle \rho \rangle$

Summary:  $\langle \vec{E} \rangle$ ,  $\langle \phi \rangle$ , and  $\langle \rho \rangle$  are related to each other just like  $\vec{E}$ ,  $\phi$ , and  $\rho$ . Since we can't say anything useful, or do any productive quantitative calculations with the real microscopic  $\vec{E}$ ,  $\phi$ , and  $\rho$  anyhow, we will never actually talk about them again. So there is no ambiguity in dropping the  $\langle \rangle$  and just writing  $\vec{E}$ ,  $\phi$ , and  $\rho$  for the spatially averaged quantities from here on.

### Capacitor



The batteries are the same, so  $\Delta\phi$  is the same, so the  $\vec{E}$ -field is the same (where in the dielectric-filled capacitor, we understand this to be the average  $\langle \vec{E} \rangle$ ).

$\Rightarrow$  the total surface charges must be the same

$Q \equiv$  charge on capacitor plates  
(this is what the battery actually supplies and what will flow off when capacitor is discharged)

$Q' \equiv$  charge that develops on surface of dielectric

$$\Rightarrow Q_0 = Q + Q'$$

again, we see that this must be bigger than  $Q_0$

Define a "dielectric constant"  $\epsilon$  such that  $Q = \epsilon Q_0$   
(so  $\epsilon > 1$ , always)

$$\Rightarrow Q' = Q_0 - Q = Q_0(1 - \epsilon)$$

We can think of the capacitor as a superposition of

vacuum capacitor

+

dielectric

$$\begin{array}{c} \text{vacuum capacitor} \\ \hline Q = \epsilon_0 Q_0 \\ \downarrow \vec{E}'' = \epsilon \vec{E} \\ \hline -Q \end{array}$$

$$\begin{array}{c} \text{dielectric} \\ \hline \vec{E}' = -4\pi \vec{P} \\ \hline \downarrow \vec{P} \end{array}$$

[Note: signs in Pwells (33) are confusing]

$$\vec{E} = \vec{E}'' + \vec{E}' = \epsilon \vec{E} - 4\pi \vec{P}$$

$$-E \hat{z} = -\epsilon E \hat{z} - 4\pi (-P \hat{z})$$

$$-E = -\epsilon E + 4\pi P$$

$$(\epsilon - 1)E = 4\pi P$$

$$\frac{P}{E} = \frac{\epsilon - 1}{4\pi}$$

$\nearrow$   $P$  is the (positive) magnitude of  $\vec{P}$

In most cases, the dielectric was sitting there without any polarization at all until we started to apply a voltage to the capacitor. In other words, our applied  $\vec{E}$  actually caused  $\vec{P}$ . So  $P/E$  is a ratio of how "susceptible" a dielectric is to polarization by an applied  $\vec{E}$ .

$$\text{define } \chi \equiv \frac{P}{E} = \frac{\epsilon - 1}{4\pi} = \text{"dielectric susceptibility"}$$

For most materials, this is a nice linear relationship:  $P = \chi E$

So the bottom line is, what is the new capacitance?

$$C = \frac{Q}{V} \quad \leftarrow \text{these are the } Q \text{ and } V \text{ that the battery actually supplies}$$

$$C = \frac{\epsilon Q_0}{V} = \epsilon C_0 \quad \text{where } C_0 = \text{vacuum capacitance} = \frac{A}{4\pi d}$$