Physics 15b (Hoffman)
Lecture #6
Thurs, Oct 4, 2007

Title: "Conductors"

Recap

- Gauss' Law + divergence thm ⇒ Maxwell #1
  $4\pi\int \rho\, dV = \oint_S \vec{E}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{E}\, dV \Rightarrow 4\pi\rho = \vec{\nabla}\cdot\vec{E}$
- $\vec{E}$ is conservative + Stokes' thm ⇒ Maxwell #2 (static case)
  $0 = \oint_C \vec{E}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} \Rightarrow \vec{\nabla}\times\vec{E} = 0$
- Laplace's eqn:
  plug $\vec{E} = -\vec{\nabla}\varphi$ into Maxwell #1
  and look only in charge-free regions ($\rho = 0$)
  $\Rightarrow \nabla^2\varphi = 0$
  Laplacian operator = "del-squared"
- Solutions to Laplace's eqn
  → smoothly "average" between boundary conditions
  → value at center of sphere = average of values over surface of sphere (no matter size of sphere)
  → can solve numerically by iteratively averaging neighboring points on a grid

Goals for today:

1. conductors vs. insulators
2. how does a conductor behave in an $\vec{E}$-field?
3. find φ in the presence of conductors: Laplace's eqn
4. uniqueness of solution
5. conductors shield $\vec{E}$-field (one way only!)
6. example: point charge near a conducting plane

Conductors vs. Insulators

We've devoted some time to computing $\vec{E}$ in various systems, that is the force per unit charge.
Now the question arises: what does a charge actually do in the presence of that force?
For a point charge q in free space, we could write $\vec{F} = q\vec{E} = m\vec{a}$ and begin to compute acceleration, etc.

But what happens when our charge is part of a material? What if we apply $\vec{E}$ to a whole lump of something composed of atoms, of nuclei and electrons? Do these individual, microscopic charges actually respond to $\vec{E}_{\text{external}}$ by moving, or are they stuck in place by their interatomic forces (which are after all just more electric forces)?

The answer is: it depends!

We can reason intuitively that for a given material, the response of the charges in that material must be proportional to the applied $\vec{E}$-field. In fact, we can write:

$$\vec{J} = \sigma\vec{E}$$

$\vec{J}$: current density = amount of charge passing through a unit cross-sectional area per unit time

$\frac{[\text{charge}]}{[\text{length}]^2[\text{time}]} = \frac{C}{m^2 s}$ or $\frac{esu}{cm^2 s}$

σ: "conductivity"

$\vec{E}$: electric field inside the material, $\frac{N}{C}$ or $\frac{dynes}{esu}$

units: $\frac{C^2}{m^2 s N} = \frac{C^2 s}{m^3 kg} = \frac{1}{\Omega\cdot m}$ [SI] (Ω: "ohm")

or $\frac{(esu)^2}{cm^2 s\cdot dyne} = \frac{1}{s}$ [cgs]

Some typical values for conductivity σ:

	$\sigma \left(\frac{1}{\Omega m}\right)$ ← sometimes written $\mho/m$
silver	6.3×10^{7}
copper	5.9×10^{7}
manganin (alloy)	2.3×10^{6}
silicon	33
salt water	23
drinking water	0.0005 - 0.05
deionized water	5.5×10^{-6}
wood	10^{-8} - 10^{-11}
glass	10^{-10} - 10^{-14}
sulfur	5×10^{-16}
rubber	10^{-13} - 10^{-16}

Conductivity σ spans 23 orders of magnitude in these materials! (Compare this to other properties of materials, like heat capacity which differ by a few orders of magnitude.)

What is σ physically? A measure of how tightly e^- are bound to their local nuclei.

metals: many e^- are free to "swim around" in an "electron sea"
semiconductors: a few e^- are thermally excited out of their bound, stuck state, to move around
solutions: some charged ions are free to move around
insulators: all e^- are bound to stationary nuclei; no charge can move

Clearly, σ is a property that can take on a continuous range of values. But it turns out that most materials have very high or very low σ. We'll simplify and consider only 2 categories of materials:

conductors: charge flows freely ("near instantaneously") in response to $\vec{E}$: $\sigma \sim \infty$

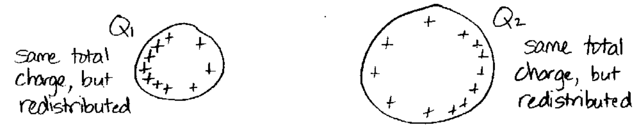
insulators: no charge flows at all : $\sigma \sim 0$

A Tale of 2 Spheres

We've considered the case of bringing 2 charged spheres together:

As we bring them close, the like charges will repel each other. If the spheres are insulators, that's the end of the story: the + charges may not like each other but they have to stay put where they started on the surface of each sphere.

But what if the spheres are conductors?
2 competing effects:
① + charges on a single sphere want to get as far away from each other as possible; intuitively they must spread out evenly over sphere
② as the spheres approach each other, the individual + charges on one sphere will repel the + charges on the other, so extra + charges will build up on the far side of each sphere:

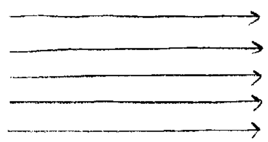


(Of course in real solids it's not the + charges that move, it's the negative e^-, but the effect of moving e^- is that the net positive charge redistributes.)

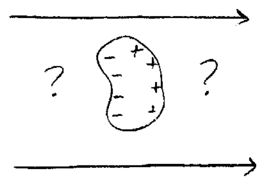
So the point is that in an applied $\vec{E}$-field, the charges in a conductor move in response to that $\vec{E}$-field.

In an idealized conductor, the charges respond to the $\vec{E}$-field near-instantaneously. So let's just consider the conductor a split second after we've turned on the $\vec{E}$-field, when the charges have already finished redistributing and are now stationary again.

Let's consider a uniform static $\vec{E}$-field:
(how would you make such a field?)



and let's plop an uncharged conductor down in it



The charges will respond to the $\vec{E}$-field and reconfigure. Furthermore, the $\vec{E}$-field itself will be modified in response to the reconfigured charges! How do we get a handle on this?

Claim:

① $\vec{E}$ must vanish inside the conductor

Why? Well, if $\vec{E}$ didn't vanish, then charges would still be moving in response to $\vec{E}$. So just wait another split second until they stop. What if they don't stop? Well then again we've used a static $\vec{E}$ to create a perpetual motion machine $\Rightarrow$ send it to the patent office.

Conclusion: after charges stop moving (near-instantaneously) $\vec{E}=0$ inside a conductor

What about at the surface? This leads us to ...

② $\vec{E}$ is perpendicular to the surface of a conductor

Why? Same reason as ①; if we had a component of $\vec{E}$ parallel to the surface, charges would still be moving.

③ φ is constant throughout a conductor, including the surface

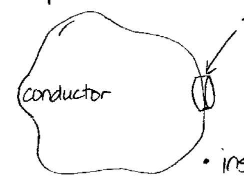
Why? $\vec{E}=-\vec{\nabla}\varphi$ so if $\vec{E}=0$ inside then we'd better have φ constant inside. And if $\vec{E}_{\parallel}=0$ on surface, then φ had better not be changing as we run around on the surface.

④ $\rho=0$ inside a conductor

Why? Because $\vec{\nabla}\cdot\vec{E}=4\pi\rho$, so if $\vec{E}=0$ then $\rho=0$

⑤ At any point *just* outside the conductor, $\vec{E}=4\pi\sigma$, oriented perpendicular to the surface of the conductor, where σ is the local surface charge.

Why? To a point *just* outside the surface, it looks like a plane with surface charge σ.



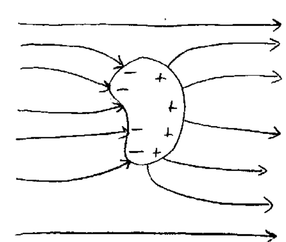
Take a small Gaussian box, area A

Apply Gauss' Law:

$$\oint_S \vec{E}\cdot d\vec{a} = 4\pi\, q_{enclosed} = 4\pi a$$

- inside surface of Gaussian box: $E=0$
- side walls of Gaussian box: $E_{\parallel}$ on surface of conductor is zero, so there is no component of E actually going out through the side walls of the Gaussian box
- outside surface of Gaussian box:
  $EA=4\pi\sigma A$
  $\Rightarrow E=4\pi\sigma$

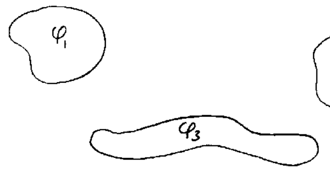
(we already knew that $\vec{E}$ was perpendicular to this outside surface)



Far from the conductor, $\vec{E}$ is unchanged but close to it, the $\vec{E}$ field lines are bent in to be perpendicular.

Laplace's eqn again:

Now let's consider several conductors, each held at a potential φ_i (e.g. by a battery or power supply)



take $\varphi=0$ at ∞

Everywhere away from these conductors, $\rho=0$ so we can apply Laplace's eqn: $\nabla^2\varphi=0$

It turns out Laplace's eqn is pretty hard to solve analytically, even for apparently simple geometries like 2 spherical conductors

We can assume there is a solution, because there's nothing stopping us from hooking up our batteries to our conductors, and then the charges will have to do *something*.

Let's prove the solution is unique.

We start by assuming there are 2 solutions $\varphi(x,y,z)$ and $\psi(x,y,z)$ which both satisfy Laplace's eqn for the given boundary conditions.

But Laplace's eqn is linear, so any linear combination of φ and ψ must also be a solution. In particular,

$$W(x,y,z) = \varphi(x,y,z) - \psi(x,y,z)$$

must be a solution. But the values of φ and ψ are fixed at the boundaries by the conductor configuration we started with at the top of the page.

$\Rightarrow W(x,y,z)$ must be zero at the boundaries

$W(x,y,z)$ still solves Laplace's eqn, but with different boundary conditions.

So $W(x,y,z)$ still obeys this value-at-center = average-on-sphere property of all solutions of Laplace's eqn.

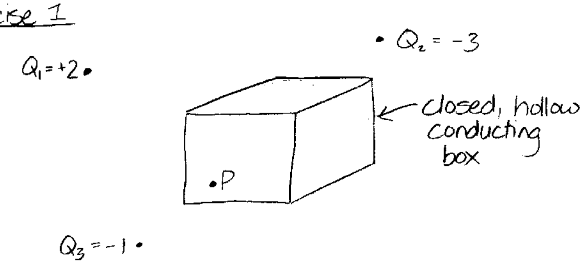
$\Rightarrow W(x,y,z)$ can have no local minimum or local maximum.

So if $W(x,y,z)$ is zero on all boundaries (including infinity) then it must be zero everywhere.

$$\Rightarrow \varphi(x,y,z) = \psi(x,y,z)$$

OK, so we proved that the solution to $\nabla^2\varphi=0$ is unique as long as φ is fixed at all boundaries (including infinity, if applicable).

Exercise 1



What is $\vec{E}$ at point P, inside the box? Why?

Answer: We know φ is constant on the surface of the box, because it is a conductor.

$\Rightarrow \varphi_{inside} = \varphi_{surface} = \text{constant}$. Why? 2 different explanations

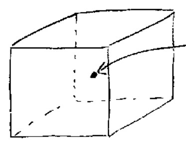
① φ can't have a local min or max inside box

② Clearly $\varphi_{in} = \varphi_{surf} = \text{constant}$ is one solution to $\nabla^2\varphi=0$ so it must be the only solution

$\Rightarrow \vec{E} = -\vec{\nabla}\varphi = 0$ inside

Exercise 2

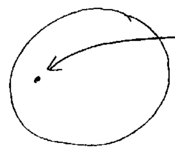
What about this:



Q at center of hollow conducting box (box itself is uncharged)

Answer: $\vec{E} \neq 0$ outside (apply Gauss' law)
not easy to calculate the exact form of $\vec{E}$.

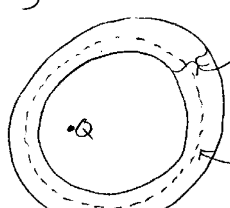
Exercise 3



Q off-center in hollow conducting sphere (sphere itself is uncharged)

Answer: $\vec{E} = \frac{Q}{r^2}\hat{r}$ outside, as if Q were at the center.
Why?

The conducting spherical shell must have some thickness



this is our conducting shell, drawn with exaggerated thickness

draw a Gaussian surface: a concentric sphere within the shell

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enclosed}$$

but $\vec{E} = 0$ because it's within the conductor
$\Rightarrow q_{enclosed} = 0$
$\Rightarrow -Q$ must have distributed itself on the inside of the shell, leaving $+Q$ to distribute itself on the outside

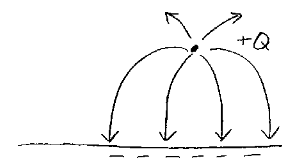
But all that outside $+Q$ knows is that the field immediately adjacent to it (just inside the shell) is $\vec{E} = 0$, so $+Q$ will distribute itself evenly.

Why didn't the same logic work for the cube?
Well it's true we will still have $-Q$ on the inner surface of the cube, leaving $+Q$ to distribute itself at will on the outer surface.
But the cube doesn't have the symmetry of the sphere, so the charge won't distribute isotropically.
In particular, it all wants to get as far away from itself as possible, so it will build up at the corners.

Image charges: a cute mathematical trick

Example: bring a point charge $+Q$ near a conducting plane; how does the charge redistribute in the plane, and what will be the resultant $\vec{E}$-field?

Qualitatively:

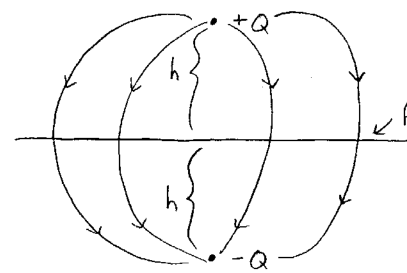


$\vec{E}$ close to $+Q$ looks just like $+Q$ alone (radial)

$\vec{E}$ perpendicular to surface

build up negative charge

But how to solve quantitatively?
Answer: apply a trick, method of images
(This method is generally useful, not some one-time brain real estate waster.)

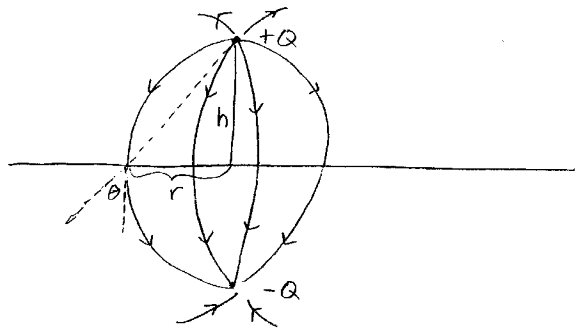


field lines are everywhere perpendicular to the middle plane

In the upper half plane, the field meets all the requirements of the actual problem:
$\vec{E}$ is perpendicular to plane of conductor
looks like point charge near $+Q$

By the uniqueness thm, if this is one solution in the upper half space, it must be the only solution.

(Note: slightly different boundary conditions than the uniqueness thm we proved: here we know $\varphi = 0$ on plane and at infinity, but instead of knowing φ by the point charge, we know the value of the charge. There's an analogous uniqueness thm for this situation → maybe prove for homework?)



$\vec{E}$ just above plane:

$$E_z = \frac{-2Q}{r^2+h^2}\cos\theta = \frac{-2Q}{r^2+h^2}\cdot\frac{h}{(r^2+h^2)^{1/2}} = \frac{-2Qh}{(r^2+h^2)^{3/2}}$$

This tells us the surface charge density:

$$\sigma = \frac{E_z}{4\pi} = \frac{-Qh}{2\pi(r^2+h^2)^{3/2}}$$

What is the total surface charge?

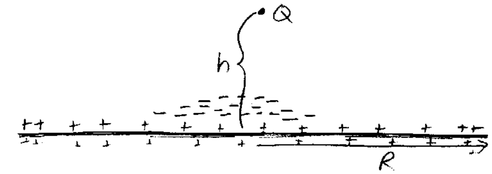
$$Q_{surf} = \int_0^\infty \sigma\, 2\pi r\,dr = -Q\int_0^\infty \frac{hr}{(r^2+h^2)^{3/2}}\,dr = -Q$$

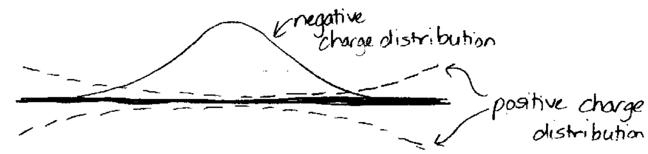
(hint: if you're too lazy or can't remember how to do an integral, go to

http://www.integrals.com

But what if our conductor started uncharged? How can it now have charge $-Q$?

There must be an equal and opposite $+Q$ distributed in plane





Closing thought: what is the force between the point charge and conducting plane?