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Physics 15b (Hoffman)
Lecture #2
Thurs, Sept 20, 2007

Title: "Gauss's Law"

Recap:

Coulomb's law: gives the force between 2 point charges

$$\vec{F}_1 = \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$$

force acting on charge #1

vector pointing from charge #2 to charge #1

3 properties of a system of charges:

U = electrostatic energy (scalar property of whole system)

$$= \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}}$$

$\vec{E}(x,y,z)$ = electric field (vector function of position)

$$= \sum_{j=1}^{N}\frac{q_j}{r_{oj}^2}\hat{r}_{oj} = \int\frac{\rho(x',y',z')\,dx\,dy\,dz}{r^2}\hat{r}$$

$\rho(x,y,z)$ = charge density (scalar function of position)

$$dq = \rho(x,y,z)\,dx\,dy\,dz$$

Goals for today:

① Define electrical flux
② Flux concept leads to a new theorem, Gauss' law which gives us a simpler way to compute $\vec{E}$ - fields of continuous charge distributions than the messy 3-dim integral we saw last time.
③ Examples: sphere, line, plane
④ Energy of an electric field

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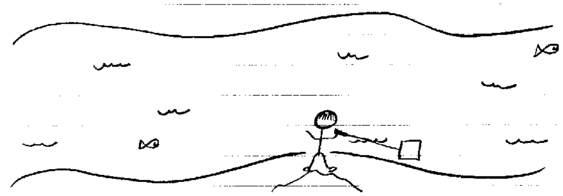
Flux:

Consider a river whose flow rate is constant in time. The velocity of water differs at different points where the river may be wider, deeper, shallower, etc. But for a given local point, the velocity is constant in time.

Now suppose we have a test loop we can dip:



We can measure the amount of water flowing through the loop per unit time. This is called the flux of water.

The flux will change as we change the location of the loop (e.g. from shallower to deeper).

The flux will also change as we change the orientation of the loop, from perpendicular (high flux) to parallel (zero flux) to the direction of the velocity.

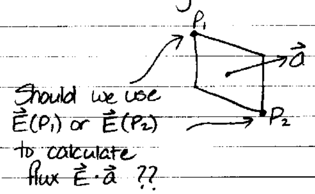
We define an area vector for the loop, $\vec{a}$, such that the magnitude of $\vec{a}$ is the area of the loop, and the direction of $\vec{a}$ is perpendicular to the loop.

$\Rightarrow$ flux = volume H_2O/time = $\vec{v}(x,y,z)\cdot\vec{a}$

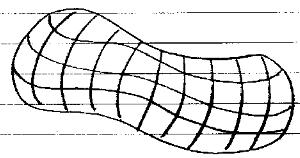
Now imagine that the $\vec{E}$-field lines we drew represent a flow field like the water. We can similarly define an electrical flux as: $\vec{E}\cdot\vec{a}$

where $\vec{a}$ is the vector representation of a patch of area. You can picture this as the # of electric field lines passing through the area represented by $\vec{a}$.

Note that if $\vec{E}$ is varying rapidly across the area represented by $\vec{a}$, then our definition of flux $\vec{E}\cdot\vec{a}$ is ambiguous:



To get around this problem, we can take a given surface S



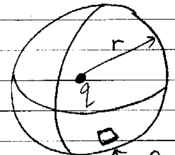
and divide it up into infinitely many patches, each of which are infinitesimally small.

Then the flux through the whole surface is:

$$\Phi = \lim_{|a_j|\to 0} \sum_{\text{all } j} \vec{E}_j \cdot \vec{a}_j = \int_{\substack{\text{entire}\\\text{surface}}} \vec{E}\cdot d\vec{a}$$

Whoa! Surface integral looks impossible to compute for the peanut above!

Don't panic: let's start with a sphere as an example



charge q at center of sphere of radius r

infinitesimal patch of area on sphere
$\vec{a}$ for this area points radially out
so does $\vec{E}$: $\vec{E} = \frac{q}{r^2}\hat{r}$ at each point
$\vec{E}$ is the same for every single $\vec{a}$

$$\Phi = \int_S \vec{E}\cdot d\vec{a} = \int_S E\, da = \underset{\frac{q^2}{r}}{E} \underbrace{\int_S da}_{\text{area of sphere} = 4\pi r^2}$$

$\Rightarrow \Phi = 4\pi q$

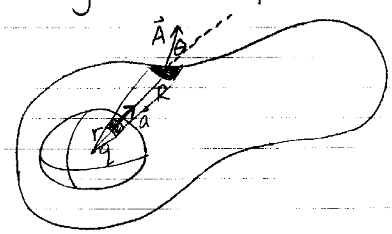
Note this doesn't depend on the size of sphere! In fact, it doesn't even depend on the shape - it could very well be a peanut, not a sphere.

How can this be true?

Intuitively: return to the water example

If q were a water source, and our sphere or peanut were a net completely enclosing it, then clearly the shape of the net wouldn't affect how much water was leaking out; the leak rate would be determined by the faucet q regardless of shape.

Mathematically: embed a sphere inside the peanut



Consider the cone extending from inner patch to outer

Flux through inner patch: $\vec{E}(r)\cdot\vec{a} = \frac{q}{r^2}a$

Flux through outer patch: $\vec{E}(R)\cdot\vec{A} = \frac{q}{R^2} A\cos\theta$

but A is bigger than a by a factor R^2/r^2 and by $1/\cos\theta$ (if it's tilted by a larger angle θ, $\cos\theta$ is smaller, $1/\cos\theta$ is bigger, and A is bigger)

$$\Rightarrow \vec{E}(R)\cdot\vec{A} = \frac{q}{R^2}\left[a\frac{R^2}{r^2}\frac{1}{\cos\theta}\right]\cos\theta = \frac{q}{r^2}a = \vec{E}(r)\cdot\vec{a}$$

⇒ Size & shape of surface doesn't matter!

What property(s) of $\vec{E}$ does this nice coincidence rely on?

The area changes in proportion to (distance)^2

In order to cancel this, so that overall shape doesn't matter, we need $\vec{E}$ to be proportional to $\frac{1}{(\text{distance})^2}$

⇒ Because $\vec{E}$ obeys an inverse square law, the flux of $\vec{E}$ through any surface is independent of the size & shape of the surface.

Gauss' law

The flux of the electric field $\vec{E}$ through any closed surface, that is the integral $\int_S \vec{E}\cdot d\vec{a}$, equals 4π times the total charge enclosed by the surface:

$$\int_S \vec{E}\cdot d\vec{a} = 4\pi \sum_i q_i = 4\pi \int \rho(x,y,z)\,dx\,dy\,dz$$

[Holds for any vector field that obeys an inverse square law. So this theorem contains both math (area $\propto$ distance^2) and physics ($\vec{E}$-field falls off like $1/\text{distance}^2$)]

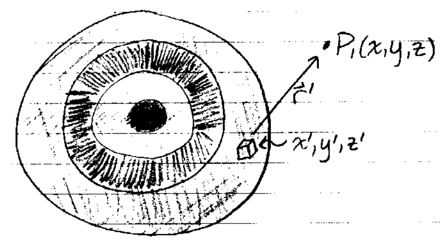
What good is Gauss' law?

Coulomb's law tells us how to compute $\vec{E}$ from q's

Gauss' law tells us, given $\vec{E}(x,y,z)$, how much charge is in any region.

Examples: spheres, lines, & planes

Sphere



Suppose $\rho(r)$ is spherically symmetric (depends only on r)

How do we compute the field at P_1?

method 1: Coulomb's law

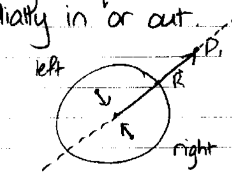
$$\vec{E} = \int_{\text{sphere}} \frac{\rho(x',y',z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}\,dx'dy'dz'\,\hat{r}'$$

integrate over spherical volume → what a MESS!

method 2: use symmetry & Gauss' law

① Which way does field point at P_1?

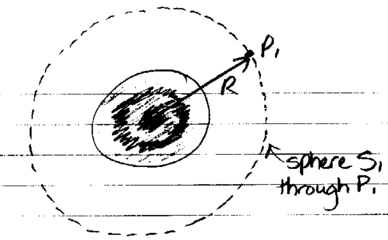
The only unique direction allowed by symmetry is radially in or out.



Any force from the left half of the sphere which is not directed along $\vec{R}$ will be cancelled by opposite component from right half of sphere.

Also, $|\vec{E}|$ must be the same at all points at radius R.

② Draw an appropriate surface through P_1



③ Apply Gauss' Law:

$$\int_{S_1} \vec{E} \cdot d\vec{a} = 4\pi \,(\text{charge enclosed in } S_1)$$

$\vec{E}$ is parallel to $d\vec{a}$
everywhere on S_1
$\Rightarrow$ eliminate vectors

$$\int_{S_1} |\vec{E}|\, da = 4\pi \,(\text{charge enclosed in } S_1)$$

$|\vec{E}|$ is constant on S_1
b/c R is constant
$\Rightarrow$ pull out of integral

$$E \int_{S_1} da = E\,(4\pi R^2) = 4\pi \,(\text{charge enclosed in } S_1)$$

$$\Rightarrow E = \frac{\text{charge inside } S_1}{R^2}$$

Whoa! E field is the same as if all charge
were concentrated at a point in the center!

Break exercise:

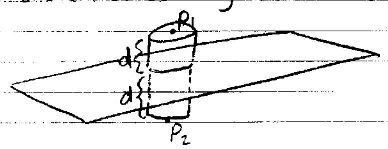
What is the electric field a distance R from a
line of continuous charge density λ (=charge/unit length)

Plane:

What's the field due to a uniformly charged plane with surface charge σ?

① symmetry: $\vec{E}$ must be perp to plane, equal & opposite on both sides

② draw an area: cylinder of radius r, length $2d$



③ Gauss' law:

$$\int_S \vec{E}\cdot d\vec{a} = 4\pi\,(\text{charge enclosed})$$

$\vec{E} \perp d\vec{a}$ on tube walls of cylinder
$\vec{E} \parallel d\vec{a}$ on endcaps

$$\int_S \vec{E}\cdot d\vec{a} = 2E(\pi r^2) = 4\pi(\pi r^2 \sigma)$$

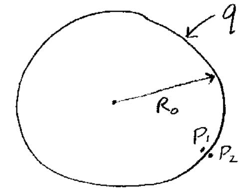
$$\Rightarrow E = 2\pi\sigma$$

Energy associated with the electric field

Last class, we learned how to write the energy of a system of charges as a sum, or integral over the charges themselves.

Now we'll suggest (without proof) a way to write the energy of a system as an integral over the $\vec{E}$-field throughout the volume of the system.

We'll start from the specific example of a spherical shell of charge (hollow in the middle). This sphere must have positive energy (it took work to bring these like charges together & assemble). In fact, it would take even more energy dU to compress the shell further from initial radius R_0 to final radius $R_0 - dr$. We'll compute this incremental energy.



q = total charge on surface

$\sigma = \dfrac{q}{4\pi R_0^2}$

Inside: $\vec{E} = 0$ (no charge enclosed)
Outside: $\vec{E} = \dfrac{q}{r^2}\hat{r}$ (as if charge were concentrated at center)

P_1 (immediately inside) : $\vec{E} = 0$

P_2 (immediately outside) : $\vec{E} = \dfrac{q}{R_0^2}\hat{R}_0$

$$= 4\pi\left(\frac{q}{4\pi R_0^2}\right)\hat{R}_0 = 4\pi\sigma\,\hat{R}_0$$

Good, if we're very close to the sphere, it looks like a plane (like us standing on earth) and therefore the field discontinuity across the surface must be $4\pi\sigma$, just as we found in the previous plane example.

Now if we compress the sphere by an infinitesimal increment dr, we'll be creating a new field $E = \dfrac{q}{R_0^2}$ over an infinitesimally thin spherical shell of volume $dV = 4\pi R_0^2 dr$ just inside the original sphere of charge. In this volume dV, the field used to be zero (no charge enclosed before compression) and is now $E = q/R_0^2$ (because the whole charge q is now enclosed after compression)

Let's compute the energy required to perform the compression - this will tell use the energy we just added by creating this new shell of non-zero field.

To compute the energy U, we need to know the work

$$W = \text{force} \cdot \text{distance}$$

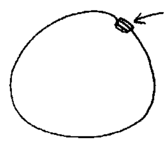
$\uparrow$ what is this? $\qquad \uparrow -dr$

(negative, b/c pointing inwards)

Force = (charge to be compressed)
$\cdot$(field against which you're compressing it)

OK, so the question boils down to:

Just what is the force on a piece of our charged balloon due to the balloon itself?



← what is the force on this due to rest of balloon?

$dq = \sigma da$

$\vec{F} = dq \cdot \vec{E}_{mystery}$

what field do we use: the field immediately inside or outside?

Answer: we use the <u>average</u> of the field just on one side and just on the other side of the plane of charge.
(If you're bored, prove it. Proof on p30 of Purcell)

⇒ Total force to compress balloon

$$= \int_S dq \cdot \frac{1}{2}(4\pi\sigma) = \int_S 2\pi\sigma \cdot \sigma da = 2\pi\sigma^2 \int_S da$$

$$= 2\pi\sigma^2\, 4\pi R_0^2 = \frac{1}{2}(4\pi\sigma)^2 R_0^2$$

⇒ Total work to compress the balloon

$$W = \frac{1}{2}(4\pi\sigma)^2 R_0^2\, dr$$

⇒ Total energy increase of system

$$dU = \frac{1}{2}(4\pi\sigma)^2 R_0^2\, dr$$

$$= \frac{1}{2}E^2 R_0^2\, dr$$

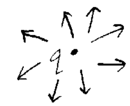
$$\frac{dU}{dV} = \frac{\frac{1}{2}E^2 R_0^2\, dr}{4\pi R_0^2\, dr} = \frac{E^2}{8\pi}$$

Suggests that the energy density of an electric field is $\frac{E^2}{8\pi}$

⇒ the energy of an entire system can be computed as

$$U = \frac{1}{8\pi}\int_{\text{Volume}} E^2 dV$$

What about a point charge?



$$\vec{E} = \frac{q}{r^2}\hat{r}$$

$$U = \int_0^\infty \frac{1}{8\pi}\underbrace{\left(\frac{q}{r^2}\right)^2}_{E^2} \underbrace{4\pi r^2 dr}_{\text{Volume increment}}$$

$$= \frac{q^2}{2}\int_0^\infty \frac{1}{r^2}dr = \frac{q^2}{2}\left[-\frac{1}{r}\right]_0^\infty = \frac{q^2}{2}\left[-\frac{1}{\infty} + \frac{1}{0}\right]$$

↑ UH-OH!

Turns out we just have to accept point charges (electrons, protons, etc) as nature's given, and not worry about the self-energy of those charges. Instead we just worry about their interaction energy.

Consider the interaction of 2 point charges q_1 & q_2:

$$U = \frac{1}{8\pi}\int E^2 dv = \frac{1}{8\pi}\int(\vec{E}_1 + \vec{E}_2)^2 dv$$

$$= \underbrace{\frac{1}{8\pi}\int E_1^2 dv}_{\substack{\text{infinite} \\ \text{self-energy} \\ \text{of } q_1 \\ \Rightarrow \text{ignore!}}} + \underbrace{\frac{1}{8\pi}\int E_2^2 dv}_{\substack{\text{infinite} \\ \text{self-energy} \\ \text{of } q_2 \\ \Rightarrow \text{ignore}}} + \underbrace{\frac{1}{8\pi}\int \vec{E}_1 \cdot \vec{E}_2\, dv}_{\text{interaction energy}}$$

interaction energy → contains all the useful information to predict behavior

note that these 2 terms are totally independent of the locations of q_1 & q_2 anyhow so even if we could compute them, they'd tell us nothing useful about how q_1 & q_2 behave

Summary:

1. We learned a nice way to compute $\vec{E}$ in high-symmetry cases.
2. We know how to get from $\vec{E}$ back to q-distribution
3. We know how to compute U from $\vec{E}$ directly (instead of starting from q's)