Physics 15b (Hoffman)
Lecture # 5
Tues, Oct 2, 2007

Title: "Math vs. Physics"

Recap:

Math: ① gradient: $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

("slope" of a scalar field)

② divergence: $\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

("flux" of a vector field out of each point)

③ curl: $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$

("rotation" of a vector field around each point)

Divergence Theorem: pick a closed surface S

$$\int_S \vec{F}\cdot da = \int_V \vec{\nabla}\cdot\vec{F}\, dV$$

Stokes' Theorem: pick a closed loop C and any surface bounded by C

$$\int_C \vec{F}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$$

Physics: ① proved $\int_{P_1}^{P_2}\vec{E}\cdot d\vec{s}$ is single-valued (not path-dependent)

② defined a scalar function $\varphi(P_1,P_2) = \int_{P_1}^{P_2}\vec{E}\cdot d\vec{s}$

③ for a given system, pick a fixed reference P_1

④ "potential" = $\varphi(x,y,z)$ = energy per unit charge to bring a test charge from P_1 to (x,y,z)

Goals for Today: combine math & physics

① Gauss' Law (physics) + divergence thm (math) → Maxwell's equation #1 (more physics)

② special case of Maxwell's eqn in the physical situation of no charge → Laplace's equation (represents physics)

③ mathematical properties of Laplace's eqn can give us physical insights

④ electrostatic force is conservative (physics) + Stokes' thm (math) → Maxwell's equation #2 in the special case of electrostatics (more physics)

⑤ roadmap to get from $\rho \longleftrightarrow \varphi \longleftrightarrow \vec{E}$

Gauss' Law + divergence thm

$$\underbrace{4\pi q_{enclosed} = 4\pi\int_V \rho\, dV = \int_S \vec{E}\cdot d\vec{a}}_{\substack{\text{physics}\\ \text{(Gauss' Law)}}} = \int_V \vec{\nabla}\cdot\vec{E}\, dV$$

(the brace labelled "math (divergence thm)" spans $\int_S \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, dV$)

$$\Rightarrow 4\pi\int_V \rho\, dV = \int_V \vec{\nabla}\cdot\vec{E}\, dV$$

Since this is true over any volume (e.g. take a volume infinitesimally to zero around a point), we can equate integrands:

$$\boxed{4\pi\rho = \vec{\nabla}\cdot\vec{E}}$$

This is the differential form of Gauss' Law → makes it much easier to go from a known $\vec{E}$-field back to charge.

This is also the first of 4 important equations called Maxwell's equations, which succinctly describe all electromagnetic interactions.

Note: this is a local eqn; don't need to know about all space

Laplace's equation

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ describes a general physical situation: it's always true, for any charge configuration

But let's look at a special case, in a region where $\rho = 0$. This case is actually not so special (although you wouldn't guess it from looking at the problems in chapters 1 & 2 where everything deals with charge configurations). But most of all the space around us has no charge density. The field & potential in this charge free space is determined by charges elsewhere.

Now remember we defined a scalar field φ:

$$\vec{E} = -\vec{\nabla}\varphi$$

So let's rewrite: $\vec{\nabla}\cdot(-\vec{\nabla}\varphi) = 0$

$$\vec{\nabla}\cdot\vec{\nabla}\varphi = 0$$

$$\underbrace{\nabla^2}\varphi = 0$$

this operator, divergence of gradient, is called the Laplacian, usually written as ∇^2, and occasionally written as Δ, and often known as "del squared"

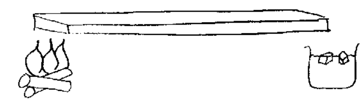
OK, as long as φ is a physical quantity (potential) then $\nabla^2\varphi = 0$ is a physical fact about the world in a charge-free region.

But we can also look at the same equation for a general scalar field $f(x,y,z)$ without physical meaning

$\boxed{\nabla^2 f = 0}$ Laplace's equation

It turns out that Laplace's equation actually describes many different physical systems, not just electrostatic potential.

For example: temperature of a system in thermal equilibrium



There is a temperature gradient across the rod, $\vec{\nabla}T$. Energy is transported (in the form of heat) from high T to low T according to

$$\text{power} \rightarrow \frac{P}{a}\hat{a} = -K\vec{\nabla}T$$

a: cross-sectional area; K: thermal conductivity

Now we can take the divergence of both sides:

$$\underbrace{\vec{\nabla}\cdot\frac{P}{a}\hat{a}} = -K\nabla^2 T$$

all along the rod except for the endpoints, this vanishes, because there are no energy sorces or sinks

$\Rightarrow \nabla^2 T = 0$ for a system in thermal equilibrium with no sources or sinks

If we know the values at the boundary, say $T_1 = 500°C$ and $T_2 = 0°C$, where there are sources and sinks of heat which fix the temperature, we can solve Laplace's eqn to get the T everywhere else.

Physical intuition: We can see that solutions to L's eqn are very smooth: they must average somehow over boundary conditions.

Laplace's eqn is also useful in fluid flow, in gravity, etc.

So it's useful to look at it in general $\nabla^2 f = 0$ and figure out its mathematical properties, which will then apply to give us physical intuition about many physical situations.

Math: if $f(x,y,z)$ satisfies $\nabla^2 f = 0$
then the average value of $f(x,y,z)$ over <u>any</u> sphere (not necessarily a small sphere!) is equal to the value of f at the center of the sphere.

Proof:

Suppose our function, $f(\mathbf{r})$, obeys Laplace's equation within some? sphere S centered at $\mathbf{r}'$:

$$\text{div grad } f = (\nabla \bullet \nabla)\, f = 0 \text{ inside S}$$

We apply the divergence theorem to the vector $f\nabla g - g\nabla f$ in the sphere with surface S excluding a tiny sphere of radius b with surface S'? having the same center. We obtain

$$\iint_S (f(\bar{n}\bullet\bar{\nabla})g - g(\bar{n}\bullet\bar{\nabla})f)dA = \iint_{S'} (f(\bar{n}\bullet\bar{\nabla})g - g(\bar{n}\bullet\bar{\nabla})f)dA + \iiint_V (f(\nabla\bullet\nabla)g - g(\nabla\bullet\nabla)f)dV$$

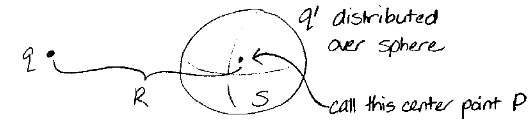
$$= \iint_S \left(\frac{f(\bar{r}'')}{4\pi a^2} + \frac{(\bar{n}\bullet\bar{\nabla})f(\bar{r}'')}{4\pi a}\right)dA'' = \frac{1}{4\pi a^2}\oint\!\!\oint_S f(\bar{r}'')dA''$$

the latter being obtained by substituting for g. The second integral on the second line vanishes here as can be seen by applying the divergence theorem again within S and noticing that the? Laplacian applied to f is 0.

The right hand side here is the average value of f on S. The similar integral over S' is evaluated in exactly the same way and is the average value of f on S'. We conclude that the average value of v on any sphere with center at r' is the same. Obviously as the sphere approaches radius 0 that average value becomes the value of f at $\mathbf{r}'$.

OK, the value-at-center = average-value-on-sphere property is <u>mathematically</u> true for any f satisfying $\nabla^2 f = 0$

How does it square with our physical intuition for φ?



If the sphere exists and we bring in charge q from infinity to a distance R, we must do work: $\frac{qq'}{R} = q'\varphi(P)$ (reference at infinity)

But if q exists and we bring in the sphere (assume the sphere was already assembled at infinity, so neglect its self-energy) then the work must be the same as q' times the average of φ due to q over surface S: $q'\varphi_{avg}$ (surface S around P)

$\Rightarrow \varphi(P) = \varphi_{avg}$(surface S around P)

This demonstration for φ relies on the fact that spherically symmetric q' behaves like it's all at point P, when q is brought in from infinity.

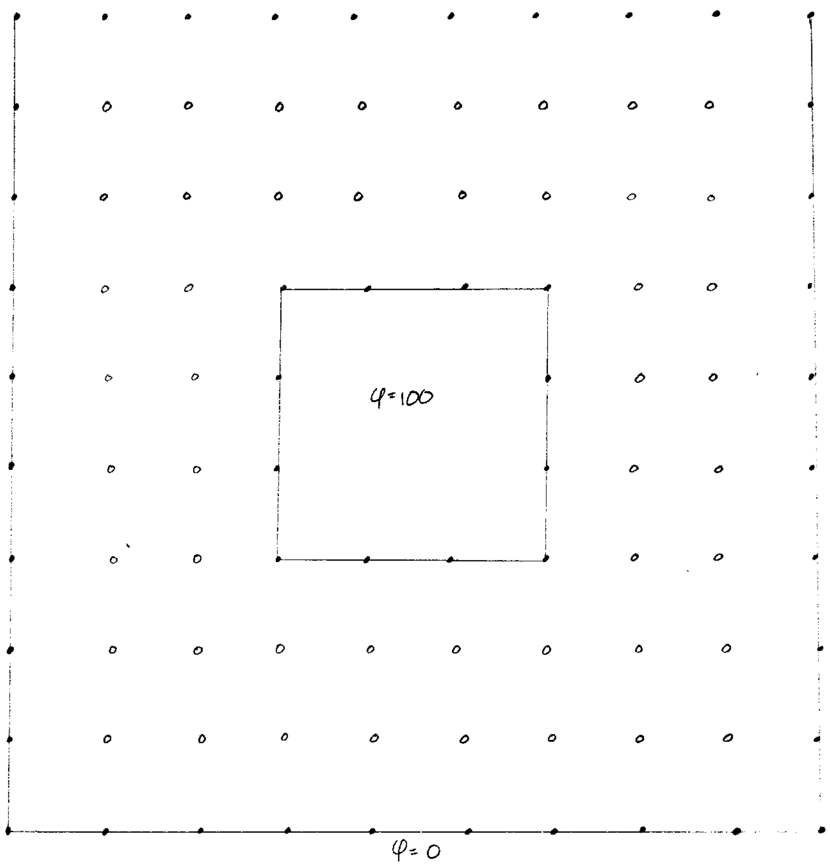
Now we've proved that $\varphi(P) = \varphi_{avg}$ (surface S around P) when φ is the potential of a point charge q. But we can use the superposition principle: if it's true for the specific φ from point charge q, then it must be true for the sum of φ's from several point charges, and it must be true for a general φ from an integration over any charge distribution outside the sphere.

OK, we prove this value-at-center = average-value-on-sphere property mathematically, so it must be true physically. We offered a redundant but more physically intuitive proof that it is true for electrostatic potentials in charge-free regions. But what good does this do us?

<u>Relaxation Method</u>: A numerical method for solving $\nabla^2\varphi = 0$ to find φ everywhere in space when it is initially known only on the boundaries.

<u>Example</u>: Suppose we have some sort of square cross section coaxial cable, and we apply $\varphi = 100V$ to the center conductor, but ground the outside $\varphi = 0$. What is the potential distribution throughout the insulator?

1. Set up a grid
2. Assign some reasonable initial values (guess)
3. Start systematically averaging 4 neighbors to get center value
4. When no values change by more than some error value (say 1%) on a given pass, then you're done



Another use of the value-at-center = average-over-sphere property of φ in a charge free region:

<u>Impossibility thm</u>: can't construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Proof: Suppose you could, i.e. there is a point P at which positive q is in stable equilibrium.
But if q is in stable equilibrium, then φ must have a local minimum at P. But local minima (& maxima) are not allowed for solutions to Laplace's eqn

<u>Electrostatic force is conservative + Stokes' thm</u>

The electrostatic force is conservative means that the work done to get q from point P_1 to P_2 in a static $\vec{E}$-field is path-independent. In particular, for a closed loop:

$$\oint_c \vec{E} \cdot ds = 0$$

because it shouldn't take any work to go nowhere

this funky circle just means integrate over a closed loop

$$\underbrace{0 = \oint_c \vec{E} \cdot d\vec{s}}_{\text{physics}} \underbrace{= \int_S \vec{\nabla} \times \vec{E} \cdot d\vec{a}}_{\text{math}}$$

$$\Rightarrow \int_S \vec{\nabla} \times \vec{E} \cdot d\vec{a} = 0$$

Since this is true for <u>any</u> surface (we can pick C arbitrarily small) we see that the integrand must vanish.

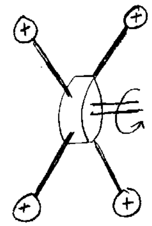
$\vec{\nabla} \times \vec{E} = 0$ everywhere for electro<u>static</u> fields

This is Maxwell's 2nd eqn in the special case of static fields.

Again, we used math to turn a non-local (hard-to-use) physics statement $\oint_c \vec{E} \cdot d\vec{s} = 0$ into a local physics statement $\vec{\nabla} \times \vec{E} = 0$

How can we get some more physical intuition for $\vec{\nabla}\times\vec{E}=0$?

Construct a "curlmeter":



But if a static $\vec{E}$ can produce a non-zero torque, then our curlmeter becomes a perpetual motion machine!

(See, impossibility thms can be quite useful!)

Summary

Gauss' Law: $\int_S \vec{E}\cdot da = 4\pi \underbrace{\int \rho\, dV}_{q_{enclosed}}$

tells us how to get from ρ to $\vec{E}$ in situations of high symmetry

differential form of Gauss' Law: $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

easier way to get from $\vec{E}$ back to ρ

electric potential: $\varphi = -\int \vec{E}\cdot d\vec{s}$

electrostatic force is conservative

for a finite charge config, can pick reference point P at infinity and define $\varphi=0$ at infinity

$\varphi = \int \frac{\rho}{r}\, dv$

if φ is integral of $\vec{E}$, then $\vec{E} = -\vec{\nabla}\varphi$

plug into differential form of Gauss' Law: $\nabla^2\varphi = -4\pi\rho$