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Physics 15b (Hoffman)

Lecture #21

Thurs, Dec 6, 2007

Title: Dielectrics again

Last time:

$$\phi(\vec{r}) = \underbrace{\frac{q_{\text{tot}}}{r}}_{\text{monopole}} + \underbrace{\frac{\hat{r} \cdot \vec{p}}{r^2}}_{\text{dipole}} + \underbrace{\frac{1}{r^3} K_2 + \dots}_{\text{quadrupole}}$$

where we can compute \vec{p} for a ^{localized} charge distribution:

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau' \rightarrow \sum_{\text{all discrete charges } q_i} q_i \vec{r}_i$$

continuous charge distrib. all discrete charges q_i

For a neutral charge distribution, our choice of origin doesn't affect the value of \vec{p} , but we typically choose the origin at the most symmetric point, just for ease of computation.

(Note: we're not often worried about calculating \vec{p} for a charged distribution, because the far field will be dominated by the $1/r$ term. But just in case we do want to carry the expansion one term further to get a little more accuracy for the far field of a charged distribution, we should pick the origin at the "center of gravity" of the charge distribution.)

Of course our choice of origin cannot affect the real, physical field and potential at a given point, in other words our choice of origin cannot affect the total sum of the infinite series expansion. But our choice of origin could shift the potential from a lower order term, where it is easy to calculate, to a higher order term, where it is more painful to calculate.)

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How the world affects a dipole:

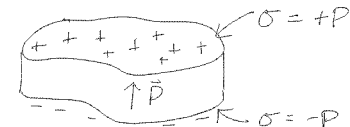
uniform \vec{E} -field: $\vec{F}_{\text{net}} = 0$
 $\vec{\tau} = \vec{p} \times \vec{E}$

non-uniform \vec{E} -field: $\vec{F}_{\text{net}} = (\vec{p} \cdot \nabla) \vec{E}$

How a dipole affects the world:

\vec{P} = polarization = measure of the dipole moment of a material per unit volume
 i.e. suppose there are N dipoles / cm^3
 and each individual dipole has strength \vec{p}
 then $\vec{P} = N\vec{p}$

A slab of uniformly polarized material produces the same field, both outside and inside, as if it just has surface charge density $\pm P$ on its 2 faces.



Fields inside matter:

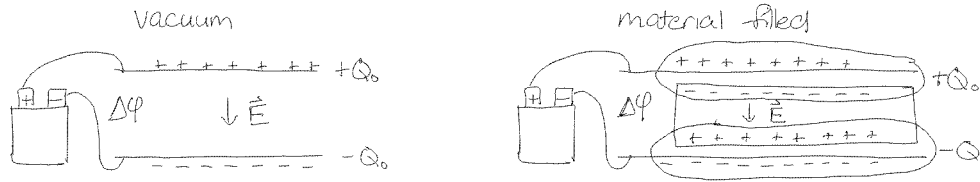
- microscopic field: messy and horrible, varies by millions of Volts on Angstrom length scale
- macroscopic field = averaged over a macroscopically small volume, denoted $\langle \vec{E} \rangle$ or sometimes just \vec{E} because we can't say anything useful about true microscopic field anyhow

We have $\langle \vec{E} \rangle$, $\langle \phi \rangle$, and $\langle \rho \rangle$

- behave just like \vec{E} , ϕ , and ρ
- often just drop the $\langle \rangle$

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Capacitor



The batteries are the same, so $\Delta\phi$ is the same, so the \vec{E} -field is the same (where in the material-filled capacitor, we understand this to be the average $\langle \vec{E} \rangle$).

⇒ the total surface charges must be the same

$Q \equiv$ charge on capacitor plates
(this is what the battery actually supplies, and what will flow off when capacitor is discharged)

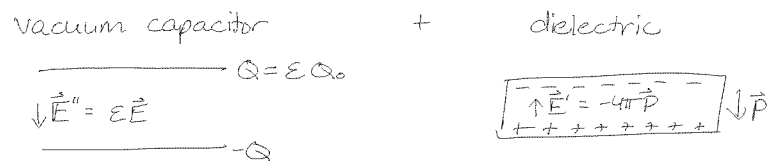
$Q' \equiv$ charge that develops on surface of material
note that Q' has the opposite sign from Q

⇒ $Q_0 = Q + Q'$
again, we see that this must be bigger than Q_0

Define a "dielectric constant" ϵ such that $Q = \epsilon Q_0$
(so $\epsilon > 1$, always)

⇒ $Q' = Q_0 - Q = Q_0(1 - \epsilon)$

We can think of the capacitor as a super-position of



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$$\vec{E} = \vec{E}'' + \vec{E}' = \epsilon \vec{E} - 4\pi \vec{P}$$

$$-E \hat{z} = -\epsilon E \hat{z} - 4\pi (-P \hat{z})$$

$$-E = -\epsilon E + 4\pi P$$

$$(\epsilon - 1)E = 4\pi P$$

$$\frac{P}{E} = \frac{\epsilon - 1}{4\pi}$$

P is the (positive) magnitude of \vec{P}

In most cases, the material was sitting there without any polarization at all until we started to apply a voltage to the capacitor. In other words, our applied \vec{E} actually caused \vec{P} . So P/E is a ratio of how "susceptible" a material is to polarization by an applied \vec{E} .

$$\text{define } \chi_e \equiv \frac{P}{E} = \frac{\epsilon - 1}{4\pi} = \text{"dielectric susceptibility"}$$

For most materials, this is a nice linear relationship: $\vec{P} = \chi_e \vec{E}$

So what is the new capacitance?

$$C = \frac{Q}{V} \leftarrow \begin{array}{l} \text{these are the } Q \text{ and } V \text{ that} \\ \text{the battery actually supplies} \end{array}$$

$$C = \frac{\epsilon Q_0}{V} = \epsilon C_0 \quad \text{where } C_0 = \text{vacuum capacitance} = \frac{A}{4\pi d}$$

Another way to think of this is that for a given applied $\pm Q$ on the capacitor plates, we would naively expect the field $E = V/d = 4\pi Q/A$. But in fact, the material partially cancels the field, so the actual total field from our battery-supplied charge $\pm Q$ is reduced:

$$E_{\text{actual}} = \frac{4\pi(Q + Q')}{A} = \frac{4\pi Q_0}{A} = \frac{4\pi(Q/\epsilon)}{A} = \frac{1}{\epsilon} \frac{4\pi Q}{A}$$

In fact, any configuration of charges Q surrounded by this material will result in a field reduced from the vacuum field by a factor $1/\epsilon$.

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What is the main assumption we have been making throughout this analysis?

We've assumed that the material responds with a polarization which is directly proportional to the ambient \vec{E} .

This is often, but not always the case.

Dielectric

\equiv an insulator which obeys $\vec{P} = \chi \vec{E}$
in other words polarization is linear with field

Why should this generally be the case for most insulators?

① non-polar materials (e.g. CH_4)

\rightarrow acquire a polarization when the applied field shifts the negative charge cloud w.r.t. positive nucleus

We already looked at a simple model for the polarizability of a single hydrogen atom:

$$\frac{\text{induced shift}}{\text{typical atomic dimension}} = \frac{\text{applied field}}{\text{typical atomic field}}$$

$$\frac{\Delta z}{a} = \frac{E}{e/a^2} \Rightarrow e\Delta z = a^3 E$$

$$\Rightarrow \vec{p} = \alpha \vec{E} \text{ with } \alpha = a^3$$

So polarizability is linear for a single atom.

Now consider a gas of molecules at atmospheric density, about 3×10^{19} molecules/cm³. But a gas is $\sim 99.9\%$ empty space, and these molecules are so far apart from each other that they are not strongly affected by dipole fields produced by their neighboring molecules.

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α = polarizability of each molecule $\Rightarrow \vec{p} = \alpha \vec{E}$
 N = # of molecules / cm³

$$\Rightarrow \vec{P} = N\vec{p} = N\alpha \vec{E}$$

$$\Rightarrow \chi_e \equiv \frac{P}{E} = N\alpha$$

\uparrow subscript "e" for electronic

b/c we will encounter a magnetic susceptibility χ_m later

For a solid, where each molecule is heavily affected by its neighbors, it's not so clear at all why the relationship typically remains linear. For now, we'll just accept that most materials, even liquids + solids, do remain linear.

② polar materials (e.g. H_2O)

\rightarrow acquire a net polarization when the applied field rotates the molecular dipoles \vec{p} along the field direction

Recall the torque on a dipole: $\vec{\tau} = \vec{p} \times \vec{E}$

Energy gained by rotating a single \vec{p} :

$$\vec{E} = E\hat{z} \uparrow \quad \begin{array}{c} \nearrow \vec{p} \\ \ominus \end{array}$$

$$\text{energy gained} = -\text{work done by } \vec{E} = -\int_0^\theta \tau d\theta$$

$$= -\int_0^\theta pE \sin\theta d\theta = -pE(-\cos\theta) \Big|_0^\theta = pE(1 - \cos\theta)$$

\Rightarrow to flip from $\theta = 90^\circ$, gain energy pE

to flip from 180° mis-aligned to perfectly aligned, gain energy $2pE$

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For a single H_2O molecule, $\vec{p} = 1.84 \times 10^{-18} \text{ esu-cm}$
 The density of water molecules is $N \sim 3 \times 10^{22} / \text{cm}^3$

\Rightarrow with complete alignment of all the dipoles in a water droplet, we would have polarization density

$$\vec{P} = N\vec{p} = 5.6 \times 10^4 \text{ esu/cm}^2$$

\Rightarrow for a polarized water droplet, we would have a field strength just outside the droplet of $\sim 10^5 \text{ statvolt/cm} = 3 \times 10^9 \text{ V/m}$

(this is $\sim 3 \times$ the breakdown field for air)

So what is really going on?

Answer: not all of the dipoles align. Thermal excitations are always causing most of the H_2O molecules to randomly orient.

This is analogous to why all of the gas molecules in the room don't settle into their lowest potential energy locations on the floor. But the air is slightly denser at the ground (which is why it's so much harder to breath at altitude).

Equipartition Theorem: predicts that every available degree of freedom will have energy $\frac{1}{2} k_B T$

where k_B = boltzmann's constant $= 1.381 \times 10^{-16} \text{ erg/K}$
 and T is a temperature in Kelvin

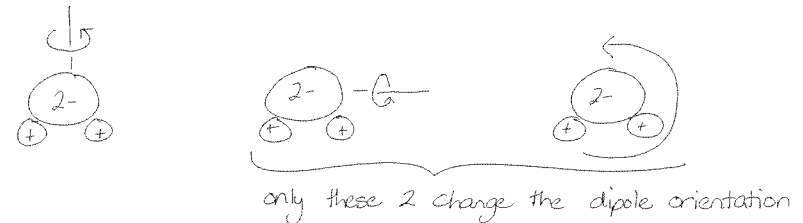
e.g. Helium atom at room temperature ($\sim 300\text{K}$) will have 3 degrees of freedom: x translational motion, y , and z , with $\frac{1}{2} k_B T$ of kinetic energy each
 \Rightarrow total translational kinetic energy of an average single He is

$$\frac{1}{2} m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} m v^2 = \frac{3}{2} k_B T$$

$$\Rightarrow v = \sqrt{\frac{3 k_B T}{m}} \approx 1.36 \text{ km/s}$$

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H_2O molecules have an additional 3 degrees of freedom from rotation:



The fractional polarization of water will be on order:

$$\frac{\text{average energy gained by rotating one molecule}}{\text{thermal energy in the relevant degrees of freedom}}$$

$$= \frac{pE}{k_B T} \leftarrow \text{on average, molecule has to turn } 90^\circ$$

\nwarrow 2 degrees of freedom

$$\Rightarrow P \approx (Np) \left(\frac{pE}{k_B T} \right) = \frac{Np^2}{k_B T} E$$

$$\chi_e = \frac{P}{E} \sim \frac{Np^2}{k_B T}$$

OK, so we've given plausible arguments that nonpolar gases and polar gases & liquids are good dielectrics, i.e. their acquired net polarization is linear in applied field.

Frequency-dependence: no material can respond to an applied field and polarize instantly (it would violate special relativity)

non-polar dielectrics: shift of charge cloud (e^-) is pretty quick
 \rightarrow cutoff frequency $\sim 10^{16} \text{ Hz}$ (adjusts in $\sim 10^{-16} \text{ s}$)

polar dielectrics: rotation of comparatively massive nuclei is slow
 \rightarrow cutoff frequency $\sim 10^4 \text{ Hz}$ (adjusts in $\sim 10^{-11} \text{ s}$)

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Gauss' Law in a dielectric

We argued in the context of the capacitor (bottom of p4) that whenever we put charged objects in a dielectric, the resultant \vec{E} -field from these charges is reduced by a factor ϵ from what it would be in vacuum.

$$\begin{aligned}\chi_e &\equiv \frac{P}{E} = \frac{\epsilon - 1}{4\pi} \\ E_{tot} &= E_{vac} + E_{pol} = E_{vac} - 4\pi P \\ &= E_{vac} - 4\pi \left(\frac{\epsilon - 1}{4\pi} E_{tot} \right) \\ E_{tot} &= E_{vac} - (\epsilon - 1) E_{tot} \\ \epsilon E_{tot} &= E_{vac}\end{aligned}$$

So suppose we drop a charge Q into a vat of oil with dielectric constant ϵ . The electric field in the oil is

$$E = \frac{Q}{\epsilon r^2}$$

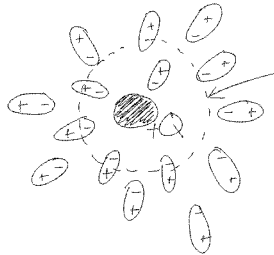
But what about Gauss' law?

$$\oint \vec{E} \cdot d\vec{a} = \frac{4\pi Q}{\epsilon}$$

sphere surrounding Q

not $4\pi Q$ as we would have had in vacuum!

Is Gauss' law violated? NO!



any sphere around Q actually contains more negative dipole ends than positive dipole ends from the surrounding material, so the actual total charge in the sphere is really reduced to Q/ϵ

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Let's make a distinction between the "free charge" Q that we intentionally placed (that came out of our battery, etc.) and the "bound charge" which comes from the dielectric response.

We could rewrite Gauss' law for our point charge above as:

$$\oint \epsilon \vec{E} \cdot d\vec{a} = 4\pi Q_{free}$$

By superposition of point charges, this holds for any free charge distribution:

$$\oint_S \epsilon \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho_{free} d\tau \Rightarrow \vec{\nabla} \cdot (\epsilon \vec{E}) = 4\pi \rho_{free} \quad (1)$$

$$\text{But Gauss' law still holds: } \vec{\nabla} \cdot \vec{E} = 4\pi \rho_{tot} = 4\pi (\rho_{free} + \rho_{bound}) \quad (2)$$

$$\text{Subtract (2) from (1): } \vec{\nabla} \cdot (\epsilon - 1) \vec{E} = -4\pi \rho_{bound}$$

$$\text{but from } \chi_e = \frac{P}{E} = \frac{\epsilon - 1}{4\pi}, \text{ we have: } \vec{\nabla} \cdot \vec{P} = -\rho_{bound} \quad (3)$$

$$\Rightarrow \vec{\nabla} \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho_{free} \quad (4)$$

At first look, (4) does not seem like progress from (1). But it turns out that (4) is always valid, even for nonlinear materials where \vec{P} is not proportional to \vec{E} .

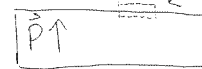
We define $\vec{D} \equiv \vec{E} + 4\pi \vec{P} =$ "displacement vector".

In uniform, linear dielectrics, $\vec{D} = \epsilon \vec{E}$

But $\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{free}$ is more generally useful.

Example What is surface charge on a uniformly polarized dielectric slab of polarization \vec{P} ?

Answer: use eqn (3) above



From this infinitesimal box at the surface, we have $\vec{P} = 0$ on top surface, \vec{P} pointing out on bottom surface, and no \vec{P} pointing in or out on sides, so $\vec{\nabla} \cdot \vec{P} = -P$

$$\Rightarrow \rho_{bound} = P \Rightarrow \sigma_{bound} \text{ on top} = P \text{ (as we already knew)}$$

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Bound charge current

Suppose we have N dipoles per cm^3 and that in time dt each changes from \vec{p} to $\vec{p} + d\vec{p}$ as each q moves by $d\vec{s}$ in time dt , i.e. $d\vec{p} = q d\vec{s}$.

$\Rightarrow \vec{P}$ changes from $N\vec{p}$ to $N\vec{p} + Nd\vec{p}$

and there is also a real current:

$$\vec{J} = Nq \frac{d\vec{s}}{dt} = N \frac{d\vec{p}}{dt} = \frac{d\vec{P}}{dt}$$

This is model-independent: changing \vec{P} always causes $\vec{J} = \frac{d\vec{P}}{dt}$

Any \vec{J} also causes \vec{B} , so we re-examine Maxwell's eqn:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}$$

We can now think of \vec{J} in 2 parts: the "free current" which we control externally, and the "bound current" which results from changing polarization in a material.

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}_{\text{free}} + \underbrace{\frac{4\pi}{c} \frac{\partial \vec{P}}{\partial t}}_{\text{bound charge current density term}}$$

But we had previously defined $\vec{D} = \vec{E} + 4\pi\vec{P}$, and noted that for a linear dielectric, $\vec{D} = \epsilon\vec{E}$.

$$\Rightarrow \vec{\nabla} \times \vec{B} = \underbrace{\frac{1}{c} \frac{\partial \vec{D}}{\partial t}}_{\text{always}} + 4\pi \underbrace{\vec{J}_{\text{free}}}_{\text{for uniform, linear materials}} = \frac{1}{c} \left(\epsilon \frac{d\vec{E}}{dt} + 4\pi \vec{J}_{\text{free}} \right)$$

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EM waves in dielectric

Start from Maxwell's equations:

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{\text{free}}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{J}_{\text{free}}$$

To simplify matters, we're not going to worry about boundary conditions for now; just assume we have an infinite expanse of uniform, linear dielectric material. In this case, we have

$$\vec{D} = \epsilon \vec{E}, \text{ and } \rho_{\text{free}} = 0 \text{ and } \vec{J}_{\text{free}} = 0$$

So Maxwell's equations reduce to:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t}$$

We can proceed the same way to eliminate \vec{B} :

$$\underbrace{\vec{\nabla} \times \vec{\nabla} \times \vec{E}}_{\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}} = -\frac{\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \nabla^2 \vec{E} = \frac{\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{Likewise, } \nabla^2 \vec{B} = \frac{\epsilon}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

These equations still have solutions of the form

$$\vec{E} = \hat{x} E_0 \sin(ky - \omega t)$$

$$\vec{B} = \hat{z} B_0 \sin(ky - \omega t)$$

But now we find that $k^2 = \frac{\epsilon}{c^2} \omega^2$

$$\Rightarrow v = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} \quad \text{so the speed of light is reduced by } 1/\sqrt{\epsilon} \text{ in a dielectric}$$