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Physics 15b (Hoffman)
Lecture #4
Thurs, Sept 27, 2007

Title: "Electric Field & Potential"

<u>Recap</u>: a lot of math, no physics

Gradient: $\vec{\nabla} f$ = vector derivative of scalar field

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

represents the direction & magnitude of steepest variation

Divergence: $\vec{\nabla}\cdot\vec{F}$ = scalar derivative of vector field

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

represents the flux into or out of each point

Curl: $\vec{\nabla}\times\vec{F}$ = vector derivative of vector field

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$$

represents the circulation around each point in the 3 orthogonal planes through the point

Divergence Thm:
choose a closed surface

$$\int_S \vec{F}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{F}\, dV$$

surface $\longleftrightarrow$ volume

Stokes' Thm:
choose a closed curve
choose <u>any</u> surface spanning that curve

$$\int_C \vec{F}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$$

contour $\longleftrightarrow$ surface

<u>Goals for Today</u>:

① Electrostatic force is conservative
② Define electric potential
③ Equipotential surfaces & field lines
④ Electric potential vs. potential energy

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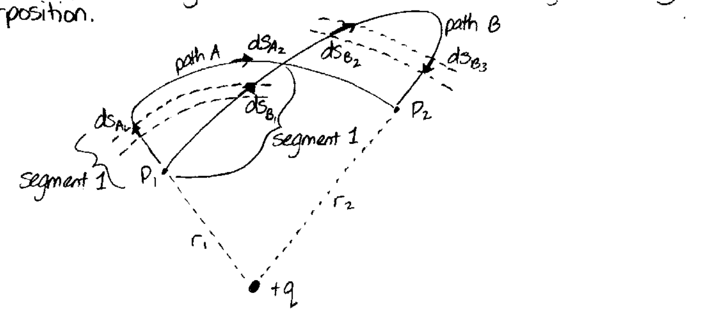
Path independence of work due to $\vec{E}$-field:

In lecture #2, we claimed w/o proof that the electrostatic force was conservative. Let's revisit that more rigorously, because we're about to make use of the conservative nature of the force to define a new concept: the electrostatic potential

Outline of proof: we'll consider the field of a single point charge, and prove that the work done to move another charge around in this point-charge-$\vec{E}$-field is path-independent. Then we apply the <u>superposition principle</u>: <u>Any</u> static $\vec{E}$-field must arise from a charge distribution which is the sum or integral over point charges. So <u>any</u> $\vec{E}$-field is just a sum or integral over point-charge-$\vec{E}$-fields. Therefore, if this path-independence is true for a single point-charge-$\vec{E}$-field, it must be true for a sum of point-charge-$\vec{E}$-fields (work done adds linearly)
$\Rightarrow$ path independence is true for <u>any</u> $\vec{E}$-field.

Superposition principle is a good trick for proving something in E&M: prove the statement for the easy-to-calculate-explicitly case of a point charge, then assert that it's always true by superposition.



$\vec{E}\cdot d\vec{s}_{A_1} = \vec{E}\cdot d\vec{s}_{B_1}$ but $\vec{E}\cdot d\vec{s}_{B_2} = -\vec{E}\cdot d\vec{s}_{B_3}$

$\Rightarrow \int_2 \vec{E}\cdot d\vec{s}_B = 0$ (cancels)

$\Rightarrow \int_1 \vec{E}\cdot d\vec{s}_A = \int_1 \vec{E}\cdot d\vec{s}_B$ and $\int_2 \vec{E}\cdot d\vec{s}_A = 0$ (perpendicular)

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OK, so $\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$ is path-independent for $\vec{E}$ from point charge.

General $\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots$ from many charges

But $\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{P_1}^{P_2} (\vec{E}_1 + \vec{E}_2 + \ldots)\cdot d\vec{s} = \int_{P_1}^{P_2} \vec{E}_1\cdot d\vec{s} + \int_{P_1}^{P_2} \vec{E}_2\cdot d\vec{s} + \ldots$

$\Rightarrow$ line integral $\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$ for electrostatic field is well-defined and single-valued even if we don't specify the path of integration

Corollary:

① What can you say about the line integral $\oint \vec{E}\cdot d\vec{s}$ around a closed loop?

② What can you say about $\vec{\nabla}\times\vec{E}$?

Electric potential

OK, if $\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$ is single-valued then it might be useful, so give it a name: electric potential

$$\Phi_{21} = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$$

↑ scalar, path independent

Why the minus sign?

If we multiply this by a test charge q:

$$q\Phi_{21} = -\int_{P_1}^{P_2} (q\vec{E})\cdot d\vec{s}$$

the force that q feels from $\vec{E}$

negative sign means this is the work you do to move q (you apply equal & opposite force $-q\vec{E}$)

therefore this is the potential energy of the test charge

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So we use the minus sign b/c it turns out to make $q\Phi$ represent a useful quantity (potential energy).

But Φ_{21} is kind of clunky b/c its a function of 2 points at the same time $\rightarrow$ it's not a field.

Potential at P_2 is only defined with respect to P_1 - just like potential energy is only defined with respect to some reference configuration.

So fix P_1 at some convenient place (often it's convenient to fix P_1 at infinity unless charges themselves extend to infinity (as they often do in textbook problems)).

Then $\Phi = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$ is a function only of P_2

$\Rightarrow$ it's a scalar field called the potential field which tells us per charge how much energy it will take to move that charge around

Example: what does the potential look like here?

$q_1 = -1$ • $(0, y_0 = 5)$

$q_2 = +2$ • $(0,0)$

Pick P_1 at infinity, then $P_2 = (x,y)$

$$\Phi(x,y) = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = -\int_{\infty}^{P_2} (\vec{E}_1 + \vec{E}_2)\cdot d\vec{s}$$

$$= -\int_{\infty}^{P_2} \vec{E}_1\cdot d\vec{s} - \int_{\infty}^{P_2} \vec{E}_2\cdot d\vec{s}$$

$$= \frac{q_1}{\sqrt{x^2 + (y-y_0)^2}} + \frac{q_2}{\sqrt{x^2+y^2}}$$

= energy required per test charge q, to bring q from ∞ to the point (x,y)

Potential is a function of position: it deals with the energy required to move a test charge around in an existing stationary configuration of charges.

Potential energy is one single property of a whole system. It has no space dependence b/c it's a property of the whole system.

From Φ back to $\vec{E}$-field

We got from $\vec{E}$-field to ϕ by $\phi = -\int_{P_1}^{P_2} \vec{E} \cdot ds$

So how do we get from ϕ back to $\vec{E}$-field?

If we go one way by integrating, it seems obvious we should go the other way by differentiating.

Intuition tells us that $\vec{\nabla}\phi = -\vec{E}$
How do we prove it?

① Taylor-expand $\phi(x,y,z)$:

$$\phi + d\phi = \phi(x,y,z) + \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\Rightarrow d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \vec{\nabla}\phi \cdot d\vec{s}$$

② But we can also get $d\phi$ by looking at one little segment of the path integral:

$$d\phi = -\vec{E} \cdot d\vec{s}$$

③ set $d\phi = d\phi$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi$$

Does the sign make sense?
Well, $\vec{\nabla}\phi$ points in the direction of increasing ϕ
so $-\vec{\nabla}\phi$ points in the direction of decreasing ϕ

A positive test charge moves in the direction of $\vec{E}$, so it "falls" from higher ϕ to lower ϕ, as expected.

Where is $\varphi = 0$?
can you draw equipotential surfaces?
field lines?

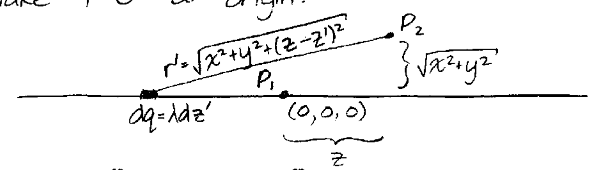
-1 •

$+2$ •

Example: infinitely long wire, charge density λ

caveat: can't take $\varphi = 0$ @ infinity b/c charges go to ∞

so take $\varphi=0$ at origin:



$$\varphi(z,r) = \int_{-\infty}^{\infty} \frac{dq}{r'} = \lambda \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$= \lambda \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{r^2+z'^2}} = \lambda \left[\ln\left(z' + \sqrt{r^2+z'^2}\right)\right]\Big|_{-\infty}^{\infty}$$

Clearly something has gone wrong!

OK, the point is that for an ∞ charge distribution, we need to be careful.

Instead, put P_1 at some distance r_1 from the wire. Since we have translational symmetry, we know that all points P at radius r_2 have same φ.

$$\varphi(P_2 \text{ at } r_2) = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = -\int_{r_1}^{r_2} \left(\frac{2\lambda}{r}\right) dr = -2\lambda \ln r + \text{const.}$$

we computed this last Thursday using Gauss' law

Go backwards from φ to $\vec{E}$, as a check:

$$\vec{E} = -\vec{\nabla}\varphi = 2\lambda \vec{\nabla}(\ln r) = 2\lambda\left(\frac{\partial(\ln r)}{\partial x}\hat{x} + \frac{\partial(\ln r)}{\partial y}\hat{y} + \frac{\partial(\ln r)}{\partial z}\hat{z}\right)$$

$$= 2\lambda\left(\frac{\partial}{\partial x}\ln\left(\sqrt{x^2+y^2}\right)\hat{x} + \frac{\partial}{\partial y}\ln\left(\sqrt{x^2+y^2}\right)\hat{y}\right)$$

$$= 2\lambda\left(\frac{\frac{2x}{2}/\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\hat{x} + \frac{\frac{2y}{2}/\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\hat{y}\right) = 2\lambda\left(\frac{x\hat{x}+y\hat{y}}{x^2+y^2}\right) = \frac{2\lambda r\hat{r}}{r^2} = \frac{2\lambda}{r}\hat{r}$$

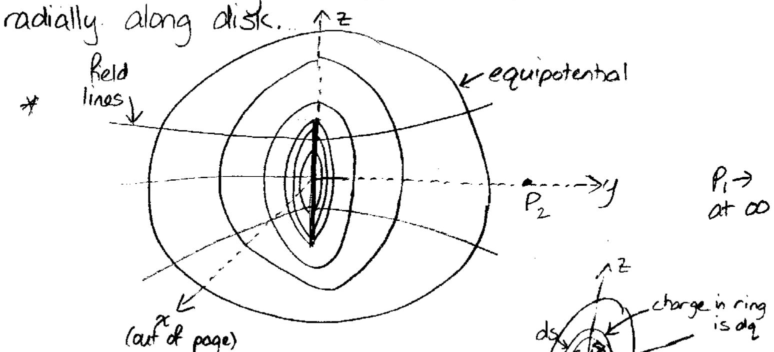
Example uniformly charged disk, radius a, charge density σ

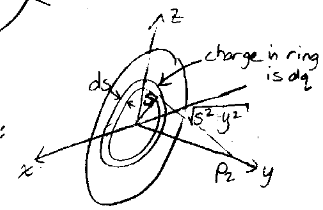
$\Rightarrow$ total charge in system $= \pi a^2 \sigma$

Far from the disk, it will look like a point charge so the equipotential surfaces will be spherical.

Close to it, it will look like a plane, so the equipotential surfaces must be parallel to surface

Potential must be largest @ center of disk and fall off radially along disk.





Let's find φ on a symmetry axis:

$$\varphi(0,y,0) = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int \frac{dq}{r}$$

$$= \int_0^a \frac{2\pi\sigma s\, ds}{\sqrt{y^2+s^2}} = 2\pi\sigma \left[\sqrt{y^2+s^2}\right]\Big|_0^a$$

$$= 2\pi\sigma\left(\sqrt{y^2+a^2} - y\right) \quad \text{for } y>0$$

$$2\pi\sigma\left(\sqrt{y^2+a^2} + y\right) \quad \text{for } y<0$$

Note: potential must be the same on both sides of disk; choose sign of $\sqrt{y^2}$ s.t. potential falls off away from positively charged disk as $|y|$ gets larger

Intuitive sign check OK ✓

We can do one more intuition check for small & large y:

small y: $\sqrt{y^2+a^2}-y$

small

$$= a\left(\sqrt{1+\frac{y^2}{a^2}}-\frac{y}{a}\right) \approx a\left(1+\cancel{\frac{y^2}{2a^2}}-\frac{y}{a}\right)$$

$$\Rightarrow \sigma = 2\pi\sigma(a-y)$$

no x-z dependence b/c equipotential is plane right next to surface

large y: $\sqrt{y^2+a^2}-y$

$$= y\left(\sqrt{1+\frac{a^2}{y^2}}-1\right) \approx y\left(1+\frac{a^2}{2y^2}-1\right) \approx \frac{a^2}{2y}$$

$$\Rightarrow \sigma \approx \frac{\pi\sigma a^2}{y}$$ like potential of point charge $\pi\sigma a^2$

* Note that field lines are perpendicular to equipotential surfaces, always.

This is true because if $\vec{E}$ has any component parallel to the equipotential surface, that would mean a non-zero force along the equipotential surface, so a non-zero work to move a charge along an <u>equi</u>-potential surface.

Important: an easy trap to fall into is to confuse the potential φ with potential energy U.

$\varphi(x,y,z)$ is a scalar field which tells the energy required to move a single test charge q around in a configuration of charges

U is a single number that gives the total energy of a configuration of charges

In lecture #2, we wrote $U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}}$

We can regroup this: $U = \frac{1}{2}\sum_{j=1}^{N} q_j \underbrace{\sum_{k\neq j}\frac{q_k}{r_{jk}}}$

this is φ at location of j^{th} charge due to all other charges labeled by k

$\Rightarrow$ generalize $U = \frac{1}{2}\int dq\,\varphi = \frac{1}{2}\int \rho\varphi\, dV$