Physics 15b (Hoffman)
Lecture #3
Tues, Sept 25, 2007

Title: "Vector Calculus"

Recap

Properties of a system of charges:

U = electrostatic energy (scalar property of whole system)

$$= \frac{1}{2}\sum_{j\neq k}\frac{q_j q_k}{r_{jk}} = \int_{\substack{\text{entire}\\\text{field}}} \frac{1}{8\pi}E^2 dV$$

↑ integrate over all volume to ∞

need to be careful to neglect self-energy terms of point charges

$\vec{E}(x,y,z)$ = electric field (vector function of position)

$$= \sum_{j=1}^{N}\frac{q_j}{r_{oj}^2}\hat{r}_{oj} = \int \frac{\rho(x',y',z')dx'dy'dz'}{r^2}\hat{r}$$

$\rho(x,y,z)$ = charge density
$\frac{1}{8\pi}E^2$ = energy density
} scalar functions of position

Flux: $\Phi = \int_{\text{surface}} \vec{E}\cdot d\vec{a}$

Gauss' law: Flux of $\vec{E}$ through any closed surface = 4π times total charge enclosed by surface

$$\underbrace{\int \vec{E}\cdot d\vec{a}}_{\text{surface integral}} = \underbrace{4\pi\int \rho\, dV}_{\text{volume integral}}$$

Goals for today:

① gradient
② div → divergence theorem
③ curl → stokes theorem

Gradient of a scalar function:

$f(x,y,z)$ = continuous, differentiable function of position (x,y,z)

How do we quantify how this function varies over space?

① One way is to list the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Reminder: $\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$

This gives us the "slope" of the function in 3 orthogonal directions.

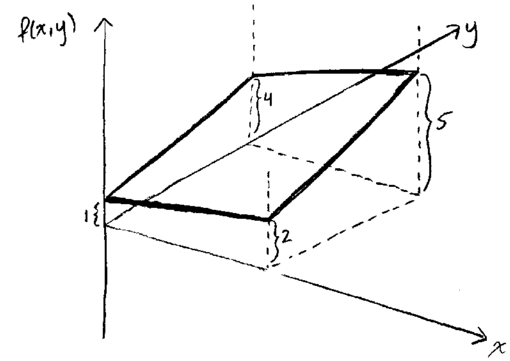
② Another way, maybe easier to visualize is just to ask in which direction the function varies fastest? And what is the slope in that direction?

So we build the vector $\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \equiv \vec{\nabla} f$

This is called the gradient of f.
f is a scalar field but its gradient $\vec{\nabla} f$ is a vector field.

The direction of $\vec{\nabla} f$ points in the direction of fastest variation, while $|\vec{\nabla} f|$ is the slope in that direction.

To see that this is true, look at an example in 2-dim:



$f(x,y) = x + 3y + 1$

$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 3$

$\vec{\nabla} f = \hat{x} + 3\hat{y}$

↑ How do we know this gives the direction of steepest slope?

Suppose the direction of steepest slope is θ from the x-axis, and the magnitude of that slope is m.

$$m = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

= total increase in f by going one unit along the unit vector $(\cos\theta, \sin\theta)$

We can maximize m by taking $\frac{dm}{d\theta} = 0$

$$\frac{dm}{d\theta} = -\sin\theta \frac{\partial f}{\partial x} + \cos\theta \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \tan\theta = \frac{\partial f/\partial y}{\partial f/\partial x}$$

$\Rightarrow \theta$ is exactly the angle of $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$

And we can plug back in to find:

$$m = \frac{\partial f/\partial x}{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2}} \frac{\partial f}{\partial x} + \frac{\partial f/\partial y}{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2}} \frac{\partial f}{\partial y}$$

$$= \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = |\vec{\nabla} f|$$

OK, so the gradient of a function $f(x,y,z)$ is $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

which represents the direction & magnitude of largest slope.

We could define gradient in a coordinate-independent way:

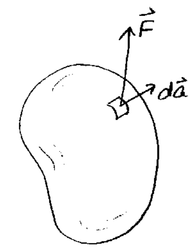
For a given direction $\hat{n}$,

$\vec{\nabla} f \cdot \hat{n}$ gives the slope of f in that direction.

Div (divergence) of a vector function

$\vec{F}(x,y,z)$ = vector function

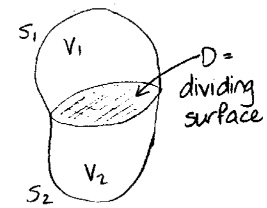
Consider a finite volume V:



We can compute the flux of $\vec{F}(x,y,z)$ through the surface of V:

$$\Phi = \int_S \vec{F} \cdot d\vec{a}$$

Now what if we subdivide V:



We can compute $\Phi_1 + \Phi_2$

$$= \int_S \vec{F} \cdot d\vec{a} + \int_D \vec{F} \cdot d\vec{a} - \int_D \vec{F} \cdot d\vec{a}$$

$$= \int_S \vec{F} \cdot d\vec{a} = \Phi$$

So the flux through any division of V and S into smaller volumes and surfaces is still equal to the flux through the original S. This is because no matter how many times we subdivide it, each dividing surface counts twice in the flux integral, but with opposite sign, because $d\vec{a}$ takes opposite sign. ("Outward" in one volume patch is "inward" in the shared surface of the adjacent volume patch.)

Flux is a macroscopic property of some surface we chose, in addition to the field $\vec{F}(x,y,z)$. We want to find some local "flux-like" property which doesn't depend on a chosen surface or volume, but only on the field.

As we subdivide V into smaller V_i's, we note that

$\Phi_i = \int_{S_i} \vec{F} \cdot d\vec{a}$ shrinks; it is not constant as $V_i \to 0$.

So Φ_i is not the local property we are looking for; it still depends on V_i.

But what if we divide by V_i?

Seems plausible (we will not prove it) that if we consider

$$\frac{\int_{S_i} \vec{F}\cdot d\vec{a}}{V_i}$$ it may approach a limit as S_i and $V_i \to 0$

We call this: $\operatorname{div}\vec{F} \equiv \lim_{V_i\to 0} \frac{1}{V_i}\int_{S_i}\vec{F}\cdot d\vec{a}$

$\operatorname{div}\vec{F}$ is the flux out of V_i per unit volume, in the limit of infinitesimal V_i. So $\operatorname{div}\vec{F}$ is truly a local property of $\vec{F}(x,y,z)$. Therefore $\operatorname{div}\vec{F}$ is a scalar field.

Divergence Theorem

Now remember that $\Phi = \Phi_1 + \Phi_2 + \ldots + \Phi_N$ when we subdivide a volume into N parts:

$$\int_S \vec{F}\cdot d\vec{a} = \underbrace{\sum_{i=1}^{N}}_{\text{as } V_i\to 0,\text{ this } \Sigma \to \int} \int_{S_i}\vec{F}\cdot d\vec{a} = \sum_{i=1}^{N} V_i \underbrace{\left[\frac{\int_{S_i}\vec{F}\cdot d\vec{a}}{V_i}\right]}_{\text{as } V_i\to 0,\text{ this }\to \operatorname{div}\vec{F}}$$

$$\Rightarrow \int_S \vec{F}\cdot d\vec{a} = \int \operatorname{div}\vec{F}\, dV$$

surface integral of F = volume integral of $\operatorname{div}\vec{F}$

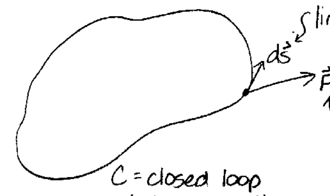
This is known as the divergence theorem, (or Gauss' thm, but this is NOT the same as Gauss' law, so we'll call it the divergence thm to avoid confusion)

Note: this is a math thm, true for any (well-behaved) vector field. Doesn't rely on physics.

Curl of a vector function

Let's look for another local property of the vector field $\vec{F}(x,y,z)$, in analogy to $\operatorname{div}\vec{F}$, by starting from the line integral around a closed loop, called the "circulation"

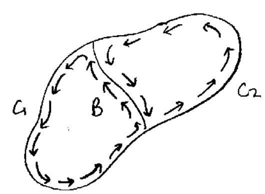
(in contrast to $\operatorname{div}\vec{F}$, where we started from the surface integral of a closed surface, called the "flux")



So in analogy to flux Φ, we write circulation Γ:

$$\Gamma = \int_C \vec{F}\cdot d\vec{s}$$

And if we subdivide C into 2 loops with a bridge B:



$$\Gamma_1 + \Gamma_2 = \int_C \vec{F}\cdot d\vec{s} + \int_B \vec{F}\cdot d\vec{s} - \int_B \vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot d\vec{s} = \Gamma$$

Because any bridge B is going to contribute once in each direction and thus cancel itself out, we can subdivide the loop indefinitely and still keep

$$\Gamma = \int_C \vec{F}\cdot d\vec{s} = \sum_{i=1}^{N}\int_{C_i}\vec{F}\cdot d\vec{s}$$

but each individual Γ_i is getting infinitesimal as we keep dividing C into smaller and smaller C_i's.

Therefore, to find a local "circulation-like" property, we need to divide by the area of the loop as we shrink it:

$$\frac{\int_{C_i} \vec{F}\cdot d\vec{s}}{a_i}$$ approaches a limit as C_i and $a_i \to 0$
(again, no proof, but seems plausible)

But unlike the V_i in the denominator of $\operatorname{div}\vec{F}$, this a_i can have an orientation. So this ratio can approach a different limit for different orientations of a_i.

So we choose a particular orientation of the area, specified by the normal vector $\hat{n}$. Then for a given point $P=(x,y,z)$ and direction $\hat{n}$ (or $\hat{x}$ or $\hat{y}$ or $\hat{z}$), we have a scalar quantity

$$\lim_{\substack{a_i\to 0\\ \text{perp. to } \hat{n}}} \frac{\int_{C_i} \vec{F}\cdot d\vec{s}}{a_i}$$

Now we can write a vector:

$$\operatorname{curl}\vec{F} = \lim_{a_x\to 0}\frac{\int_{C_i}\vec{F}\cdot d\vec{s}}{a_x}\hat{x} + \lim_{a_y\to 0}\frac{\int_{C_i}\vec{F}\cdot d\vec{s}}{a_y}\hat{y} + \lim_{a_z\to 0}\frac{\int_{C_i}\vec{F}\cdot d\vec{s}}{a_z}\hat{z}$$

Note: we didn't prove that $\operatorname{curl}\vec{F}$ is a vector; we just asserted it.

So $\operatorname{curl}\vec{F}$ is a vector function of a vector field $\vec{F}$; $\operatorname{curl}\vec{F}$ is a vector field itself.

What does $\operatorname{curl}\vec{F}$ mean?

- Its direction at each point in space is perpendicular to the plane in which the circulation of $\vec{F}$ is maximum.
- Its magnitude is the limiting value of circulation per unit area, in this plane, around the point.

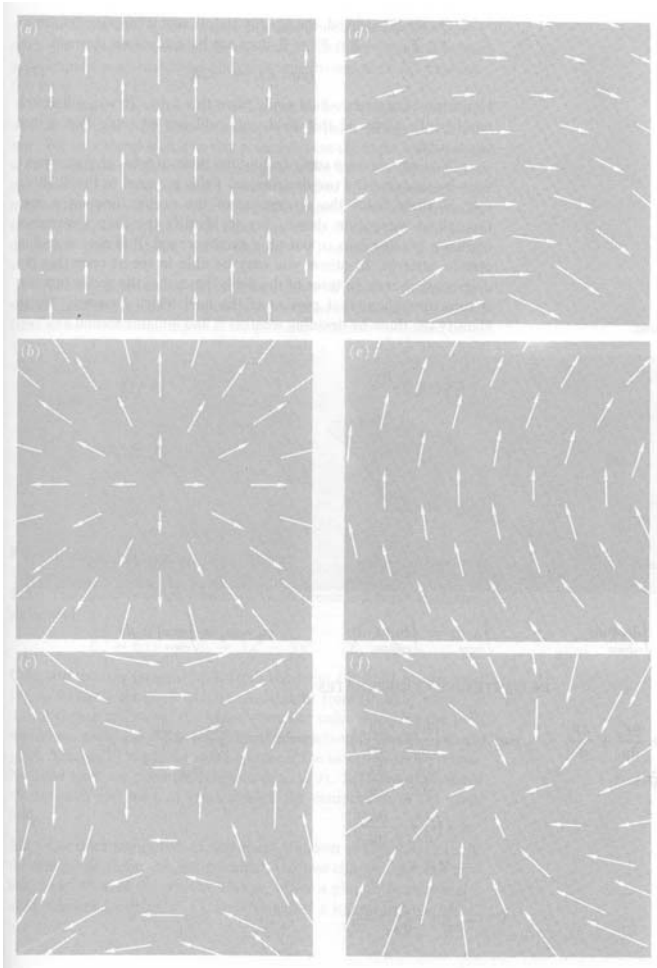
We can also define curl in a truly coordinate-independent way:

For a given direction $\hat{n}$, $(\operatorname{curl}\vec{F})\cdot\hat{n}$ gives the circulation of $\vec{F}$ about the point in the $\hat{n}$ direction.

Break exercise:

(1) Which of these vector fields have a non-zero divergence?

(2) Which have a non-zero curl?



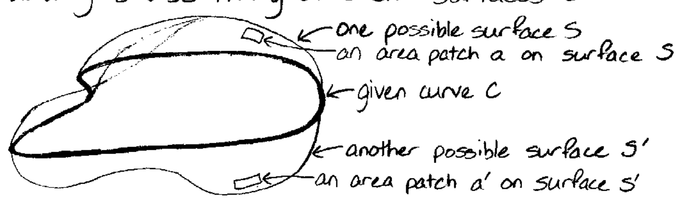
Stokes' Theorem

Remember that $\Gamma = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_N$
when we subdivide a surface into N parts:

$$\int_C \vec{F}\cdot d\vec{s} = \sum_{i=1}^{N} \int_{C_i} \vec{F}\cdot d\vec{s} = \sum_{i=1}^{N} a_i \left[\frac{\int_{C_i} \vec{F}\cdot d\vec{s}}{a_i}\right]$$

So far a_i is still finite-sized because we just have a finite N of them.
So in this step we just multiplied & divided by a_i.

Note that for a given closed loop C, we could actually choose many different surfaces S:



Note also that C doesn't need to lie in a plane; it can be any closed loop in 3-dim.

So, we start with a given C, then we choose a given S then we are going to take the limit as all the a_i's get infinitesimal, and $N \to \infty$:

$$\int_C \vec{F}\cdot d\vec{s} = \int_S d\vec{a} \underbrace{\left[\lim_{d\vec{a}\to 0} \frac{\int_{C_i} \vec{F}\cdot d\vec{s}}{|d\vec{a}|}\right]}$$

The important point is that we are taking this limit as $d\vec{a} \to 0$ with $d\vec{a}$ oriented perpendicular to the little patch of surface on S.
So this limit becomes the component of curl $\vec{F}$ which is parallel to $d\vec{a}$, perpendicular to local surface S.
$\Rightarrow$ this integrand becomes a dot product:
$d\vec{a} \cdot \text{curl}\,\vec{F}$
$\Rightarrow$ the integral is a surface integral of curl $\vec{F}$ over surface S

So we are led to Stokes' theorem:

$$\int_C \vec{F}\cdot d\vec{s} = \int_S (\text{curl}\,\vec{F})\cdot d\vec{a}$$

For a given closed loop C, the line integral around the loop is equal to the surface integral of curl $\vec{F}$ over any surface spanning the loop.

Compare Stokes' theorem to the divergence theorem:

$\int_C \vec{F}\cdot d\vec{s} = \int_S (\text{curl}\,\vec{F})\cdot d\vec{a}$	$\int_S \vec{F}\cdot d\vec{a} = \int_V \text{div}\,\vec{F}\cdot dV$
• Relates a line integral to a surface integral	• Relates a surface integral to a volume integral
• Can choose any surface S spanning the curve C	• Volume is uniquely specified by given surface S
• This theorem makes sense in 2-dim or 3-dim (2-dim version = Green's theorem)	• This theorem makes sense only in 3-dim

We've defined 3 kinds of derivatives:

Gradient: a vector derivative of a scalar field

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

direction: distance of steepest variation of f
magnitude: slope of f in that direction

Divergence: a scalar derivative of a vector field

$$\text{div}\,\vec{F} \equiv \lim_{V_i \to 0} \frac{\int_{S_i} \vec{F}\cdot d\vec{a}}{V_i}$$

a local scalar property that tells us the amount of flux emanating out of (or into) each point in space

Curl: a vector derivative of a vector field

curl $\vec{F}$ is a vector such that for a given direction $\hat{n}$

$$\lim_{a_i \to 0} \frac{\int_{C_i} \vec{F} \cdot d\vec{s}}{a_i} = (\text{curl } \vec{F}) \cdot \hat{n}$$

a_i = area perpendicular to $\hat{n}$

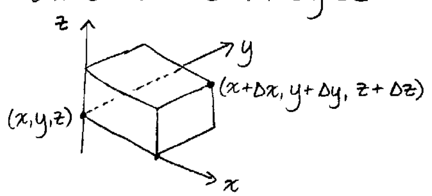
a local vector property that tells us the amount of circulation around each point in space, for each direction of circulation $\hat{n}$

direction: perpendicular to largest circulation area
magnitude: circulation in that direction

We already have a Cartesian coordinate way to calculate the gradient $\vec{\nabla} f$ so let's find a coordinate representation for div $\vec{F}$ and curl $\vec{F}$ too so we will have some practical way to actually calculate.

Div $\vec{F}$ in Cartesian

We need to take a limit as a very small volume shrinks to zero. For ease of understanding this in Cartesian coordinates, lets take our volume V to be a small box of volume $\Delta x \Delta y \Delta z$



Now we need to compute the flux of $\vec{F}$ out of this box:

$$\int_S \vec{F} \cdot d\vec{a} = \int_{top} \vec{F} \cdot d\vec{a} + \int_{bottom} \vec{F} \cdot d\vec{a} + \int_{right} \vec{F} \cdot d\vec{a} + \int_{left} \vec{F} \cdot d\vec{a}$$

$$+ \int_{back} \vec{F} \cdot d\vec{a} + \int_{front} \vec{F} \cdot d\vec{a}$$

Let's consider one of these 6 parts: $\int_{top} \vec{F} \cdot d\vec{a}$

As the box gets small, the top gets small, so to first order we can take the value of $\vec{F}$ over the top of the box to be a constant equal to its value at the center of the top:

$$\vec{F}\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}, z + \Delta z\right)$$

$$\approx \vec{F}(x,y,z) + \frac{\Delta x}{2}\frac{\partial \vec{F}}{\partial x}(x,y,z) + \frac{\Delta y}{2}\frac{\partial \vec{F}}{\partial y}(x,y,z) + \Delta z \frac{\partial \vec{F}}{\partial z}(x,y,z)$$

The flux through the top of the box is just the z-component of this vector, times the area of the top:

$$\int_{top} \vec{F} \cdot d\vec{a} = \Delta x \Delta y \left\{ F_z(x,y,z) + \frac{\Delta x}{2}\frac{\partial F_z}{\partial x}(x,y,z) + \frac{\Delta y}{2}\frac{\partial F_z}{\partial y}(x,y,z) + \Delta z \frac{\partial F_z}{\partial z}(x,y,z) \right\}$$

Likewise, the flux through the bottom is

$$\int_{bottom} \vec{F} \cdot d\vec{a} = -\Delta x \Delta y \left\{ F_z(x,y,z) + \frac{\Delta x}{2}\frac{\partial F_z}{\partial x}(x,y,z) + \frac{\Delta y}{2}\frac{\partial F_z}{\partial y}(x,y,z) \right\}$$

The net flux through top & bottom is

$$\int_{top} \vec{F} \cdot d\vec{a} + \int_{bottom} \vec{F} \cdot d\vec{a} = \Delta x \Delta y \Delta z \frac{\partial F_z}{\partial z}(x,y,z)$$

We can apply this same logic to right & left side pairs, and to front & back side pairs, to get the total flux of $\vec{F}$ out of the box:

$$\int_S \vec{F} \cdot d\vec{a} = \Delta x \Delta y \Delta z \left[\underbrace{\frac{\partial F_x}{\partial x}(x,y,z)}_{\text{right-left contribution}} + \underbrace{\frac{\partial F_y}{\partial y}(x,y,z)}_{\text{back-front contribution}} + \underbrace{\frac{\partial F_z}{\partial z}(x,y,z)}_{\text{top-bottom contribution}} \right]$$

$$\text{div } \vec{F} = \frac{\int_S \vec{F} \cdot d\vec{a}}{\Delta x \Delta y \Delta z} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

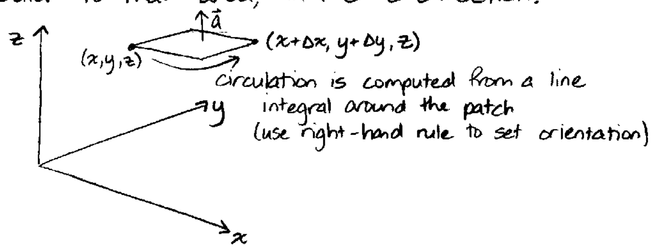
We abbreviate this with a dot product notation:

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

Curl $\vec{F}$ in Cartesian

We need to take a limit as a very small area shrinks to zero. And we need to do it 3 times, for 3 different orientations of the area, to get the 3 components of the vector curl $\vec{F}$.

Since we're looking for an expression for curl $\vec{F}$ in Cartesian coordinates, let's start with an area in the x-y plane, which will give us the component of curl $\vec{F}$ perpendicular to that area, in the $\hat{z}$ direction.



$$\oint_C \vec{F}\cdot d\vec{s} = \underbrace{\int_x^{x+\Delta x} F_x\,dx}_{\text{front}} + \underbrace{\int_y^{y+\Delta y} F_y\,dy}_{\text{right}} + \underbrace{\int_{x+\Delta x}^{x} F_x\,dx}_{\text{back}} + \underbrace{\int_{y+\Delta y}^{y} F_y\,dy}_{\text{left}}$$

Look at the right side: for a very small dy we can write $\vec{F}(x,y,z)$ as a constant taking its average value along the line segment:

$$\vec{F}(x,y,z) + \Delta x\frac{\partial \vec{F}}{\partial x}(x,y,z) + \frac{\Delta y}{2}\frac{\partial \vec{F}}{\partial y}(x,y,z)$$

So the line integral along this small $d\vec{s}$ oriented in the $\hat{y}$ direction singles out the y-component of this

$$\int_{\text{right}} \vec{F}\cdot d\vec{s} = \Delta y\left\{F_y(x,y,z) + \Delta x\frac{\partial F_y}{\partial x}(x,y,z) + \frac{\Delta y}{2}\frac{\partial F_y}{\partial y}(x,y,z)\right\}$$

Likewise, the left side gives:

$$\int_{\text{left}} \vec{F}\cdot d\vec{s} = -\Delta y\left\{F_y(x,y,z) + \frac{\Delta y}{2}\frac{\partial F_y}{\partial y}(x,y,z)\right\}$$

$$\Rightarrow \int_{\text{right}} \vec{F}\cdot d\vec{s} + \int_{\text{left}} \vec{F}\cdot d\vec{s} = \Delta x\Delta y\frac{\partial F_y}{\partial x}(x,y,z)$$

By analogy, $\displaystyle\int_{\text{back}} \vec{F}\cdot d\vec{s} + \int_{\text{front}} \vec{F}\cdot d\vec{s} = -\Delta x\Delta y\frac{\partial F_x}{\partial y}(x,y,z)$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{s} = \Delta x\Delta y\left[\frac{\partial F_y}{\partial x}(x,y,z) - \frac{\partial F_x}{\partial y}(x,y,z)\right]$$

$$\Rightarrow (\text{curl }\vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

And by further analogy,

$$\text{curl }\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

We can abbreviate this with a cross product notation:

$$\text{curl }\vec{F} = \vec{\nabla}\times\vec{F}$$

Example: compute the div and curl of the $\vec{E}$-field from a line charge oriented along $\hat{x}$-axis

Last time, we found the field from a line charge is $\vec{E} = \frac{2\lambda}{r}\hat{r}$ (outside the radius of the wire)

$$\text{div }\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{\partial}{\partial y}\left(\frac{2\lambda y}{y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{2\lambda z}{y^2+z^2}\right)$$

$$= 2\lambda\left[\left(\frac{1}{y^2+z^2} - \frac{2y^2}{(y^2+z^2)^2}\right) + \left(\frac{1}{y^2+z^2} - \frac{2z^2}{(y^2+z^2)^2}\right)\right] = 0$$

$$\text{curl }\vec{E} = \vec{\nabla}\times\vec{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{2\lambda y}{y^2+z^2} & \frac{2\lambda z}{y^2+z^2} \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$$

$$= \underbrace{\frac{\partial}{\partial x}\left(\frac{2\lambda y}{y^2+z^2}\right)}_{0}\hat{z} + \frac{\partial}{\partial y}\left(\frac{2\lambda z}{y^2+z^2}\right)\hat{x} - \underbrace{\frac{\partial}{\partial x}\left(\frac{2\lambda z}{y^2+z^2}\right)}_{0}\hat{y} - \frac{\partial}{\partial z}\left(\frac{2\lambda y}{y^2+z^2}\right)\hat{x}$$

$$= 2\lambda\left[\left(\frac{-2yz}{y^2+z^2}\right) - \left(\frac{-2yz}{y^2+z^2}\right)\right]\hat{x} = 0$$

So for the $\vec{E}$-field from a line charge (outside the wire):

both $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\times\vec{E} = 0$

<u>Summary</u>:

3 derivatives: gradient, divergence, curl

can express each one coordinate-free or in Cartesian

divergence theorem: relates surface integral to volume integral

Stoke's theorem: relates line integral to surface integral

Today: JUST MATH, no physics

Next time: Apply this math to physics

Answers to exercises on page 8 of lecture notes