Physics 153 (Hoffman)
Lecture #3
Thurs, Feb 5, 2009

Energy, Conductors

Last time: tools for computing forces exerted by stationary charges

1. Coulomb's law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$

   * superposition
   * inverse square law

2. Gauss' law
   * Integral form: $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enclosed}$
   * Differential form: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$

3. Potential

   $\vec{\nabla}\times\vec{E} = 0$ for static fields (physical fact)

   $\Rightarrow$ can define scalar $\varphi(\vec{r})$ such that $-\vec{\nabla}\varphi(\vec{r}) = \vec{E}(\vec{r})$ (math consequence)
   * Poisson's equation: $\nabla^2\varphi = \frac{-\rho}{\varepsilon_0}$
   * Laplace's equation: $\nabla^2\varphi = 0$ (in regions with no charge)

Wrap up last time:

Poisson gives us ρ in terms of φ; what we really want is φ in terms of ρ so that we can take a simple gradient of φ to find $\vec{E}$ in terms of ρ.

As usual, we're going to use superposition, so we start with a point charge q at the origin.

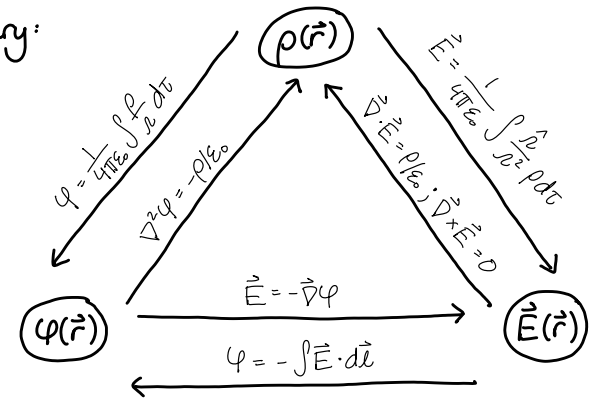
Setting $\mathcal{O}$ at infinity, we have

$$\varphi(\vec{r}) \equiv -\int_\infty^{\vec{r}} \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \frac{-q}{4\pi\varepsilon_0}\left(-\frac{1}{r'}\right)\Big|_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

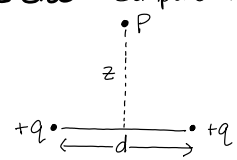
So, using superposition, we have for any charge distribution:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}d\tau'$$

Summary:



Exercise: Compute the field at P from the following charge distribution:



Goals for today:

* Energy
* Conductors (necessitates a few comments on Laplace's equation; Laplace's eqn will be covered in more detail next lecture)

Energy

We want to know how much energy is contained in a given charge distribution

For example:

$+q \bullet \quad \bullet +q$ clearly has more potential energy than: $+q \bullet \quad \bullet +q$

because we would have to do more work (put more energy into the system) to bring 2 positive q's closer together.

So we'll compute the energy of a charge distribution as the work required to assemble it all, starting with all of our charges at infinity (infinitely far apart from each other).
This amounts to choosing our zero of energy (like setting gravitational potential energy $=0$ at the floor).

Work to move a single charge Q from $\vec{a}$ to $\vec{b}$ in the presence of an arbitrary charge distribution:

$$W = \int_{\vec{a}}^{\vec{b}} \vec{F}_{you} \cdot d\vec{l} = -\int_{\vec{a}}^{\vec{b}} \vec{F}_{field} \cdot d\vec{l} = -Q \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} = Q\left[\varphi(\vec{b}) - \varphi(\vec{a})\right]$$

$\vec{F}_{you}$: force you exert to do the moving

$\vec{F}_{field}$: force exerted by the field, that you are doing work against

$$\Rightarrow \varphi(\vec{b}) - \varphi(\vec{a}) = \frac{W}{Q}$$

This gives a physical meaning to our useful scalar quantity $\varphi(\vec{r})$ (which we already knew had dimensions energy/charge).

The potential difference between $\vec{a}$ and $\vec{b}$ is the work you would have to do, per unit charge, to move a charge from $\vec{a}$ to $\vec{b}$.

If $\vec{a}$ is at infinity then $\varphi(\vec{r})$ is the work you'd have to do to bring a charge into the system, or (work-energy thm) the potential energy per charge in the system.

Collection of charges:

$q_2 \bullet$

$q_1 \bullet \qquad \bullet q_3$

$W_{tot} = W_2$ to bring in q_2 with q_1 already there
$\quad + W_3$ to bring in q_3 with q_1 & q_2 already there

By superposition, W_3 = work to bring in q_3 with q_1 already there
+ work to bring in q_3 with q_2 already there

$$\Rightarrow W_{tot} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{\mathscr{r}_{23}}$$

$$\Rightarrow W = \frac{1}{4\pi\varepsilon_0}\sum_{i>j}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

Play around with indices: goal is to get this into a form that's easier to see how we could take a continuum limit and compute energy in terms of $\vec{E}(\vec{r})$, $\varphi(\vec{r})$, or $\rho(\vec{r})$.

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i \underbrace{\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)}$$

This is the potential at $\vec{r}_i$, the position of q_i, due to <u>all</u> of the <u>other</u> charges in the system.

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i \varphi(\vec{r}_i) \xrightarrow{\text{continuum limit}} W = \frac{1}{2}\int \rho \varphi \, d\tau$$

What about writing this in terms of $\vec{E}(\vec{r})$?

Gauss' law: $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\Rightarrow W = \frac{\varepsilon_0}{2}\int (\vec{\nabla} \cdot \vec{E})\, \varphi \, d\tau$$

Integrate by parts (see chapter 1, eqn 1.59 for a review of multi-variable integration by parts)

$$\Rightarrow W = \frac{\varepsilon_0}{2}\left[-\int \vec{E}\cdot(\vec{\nabla}\varphi)\,d\tau + \oint \varphi\vec{E}\cdot d\vec{a}\right]$$

over the surface of the volume in which the charges are contained

$$= \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S \varphi\vec{E}\cdot d\vec{a}\right)$$

V = volume which contains entire charge distribution
This expression for W should be independent of our exact choice of V, as long as V contains all the charge.
What if we choose V = all space?
Then the surface gets infinitely far away from the charges.

$\varphi \propto \frac{1}{r}$ and $\vec{E} \propto \frac{1}{r^2}$ while area $\propto r^2$

Since the field $\vec{E}$ extends to all space, the first integral is getting bigger and bigger as we integrate over more and more region containing $\vec{E}$.
But evidently the surface integral is getting smaller like $1/r$ in order to compensate for the growing volume integral.

$\Rightarrow$ if we take V = all space, then the surface integral vanishes and we're left with: $\boxed{W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau}$

Questions for further thought:

① Inconsistency?

(a) $W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau$ is clearly always positive

but (b) $W = \frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{\mathscr{r}_{ij}}$ can be positive or negative, e.g. consider the charge distribution:

$+q \bullet \qquad \bullet -q$

What did we do wrong? Which is correct?

② Where is the energy stored? In the charges or the field?

Laplace's equation: $\nabla^2\varphi = 0$

*Reminder where it comes from:
Gauss's law in differential form: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$

Defined potential such that: $\vec{E} = -\vec{\nabla}\varphi$

Plug in: $\nabla^2\varphi = -\frac{1}{\varepsilon_0}\rho$

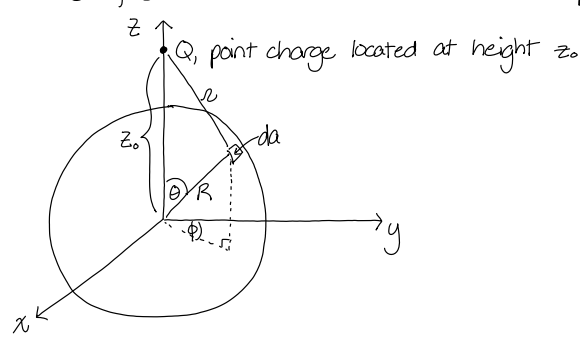
In regions where there is <u>locally</u> no charge ($\rho = 0$ locally) the potential there <u>locally</u> satisfies $\nabla^2\varphi = 0$

* Two useful properties of Laplace's equation:

① $\varphi(\vec{r}_0)$ = average of $\varphi(\vec{r})$ over sphere centered at $\vec{r}_0$

Actually, it is <u>mathematically</u> true that if any function $f(x,y,z)$ satisfies $\nabla^2 f = 0$, then the average value of $f(x,y,z)$ over <u>any</u> sphere (not necessarily a small sphere) is equal to the value of $f(x,y,z)$ at the center of the sphere.

But we will give a specific <u>physical</u> proof of this for $\varphi(\vec{r})$.
We want to prove this fact for any $\varphi(\vec{r})$ in a charge-free region... seems daunting... use the superposition principle!
Consider $\varphi(\vec{r})$ in a charge-free region where the field is produced by a single point charge Q outside this region.
Compute the average $\varphi(\vec{r})$ over some sphere in the charge-free region.
For convenience, choose coordinates centered at the sphere's center.



At any point on the surface of the sphere,

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\mathscr{r}} \quad \text{where } \mathscr{r} = \sqrt{z_0^2 + R^2 - 2z_0R\cos\theta}$$

Compute average φ over sphere's surface:

$$\varphi_{avg} = \frac{1}{4\pi R^2}\oint_S \varphi\, da$$

$$= \frac{1}{4\pi R^2}\frac{Q}{4\pi\varepsilon_0}\int_0^{\pi} R\,d\theta \int_0^{2\pi} R\sin\theta\, d\phi \frac{1}{\sqrt{z_0^2 + R^2 - 2z_0R\cos\theta}}$$

$$= \frac{Q}{8\pi\varepsilon_0}\int_0^{\pi}\frac{\sin\theta\, d\theta}{\sqrt{z_0^2 + R^2 - 2z_0R\cos\theta}}$$

$$= \frac{Q}{8\pi\varepsilon_0}\left[\frac{2}{2z_0R}\sqrt{z_0^2 + R^2 - 2z_0R\cos\theta}\right]\Bigg|_0^{\pi}$$

$$= \frac{Q}{8\pi\varepsilon_0 z_0 R}\left[(z_0+R) - (z_0-R)\right] = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z_0}$$

But this is exactly $\varphi(\vec{r})$ at the origin (center of sphere).

By superposition, this will hold true for any charge distribution outside the sphere.

Corollary: $\varphi(\vec{r})$ can have no local maxima or minima

② Given $\varphi(\vec{r})$ on closed surface, $\varphi(\vec{r})$ is <u>uniquely</u> determined inside

Proof: suppose there are 2 functions $\varphi_1(\vec{r})$ and $\varphi_2(\vec{r})$ which take the same values on the surface S, and which both satisfy Laplace's eqn inside S.

$$\nabla^2\varphi_1 = 0;\ \nabla^2\varphi_2 = 0 \Rightarrow \nabla^2(\varphi_1 - \varphi_2) = 0 \text{ inside}$$

But $\varphi_1 - \varphi_2 = 0$ everywhere on S (φ_1 and φ_2 have same values) and $\varphi_1 - \varphi_2$ satisfies Laplace's eqn inside

$\Rightarrow \varphi_1 - \varphi_2$ can have no maxima or minima inside

$\Rightarrow \varphi_1 - \varphi_2 = 0$ everywhere inside

$\Rightarrow \varphi_1(\vec{r}) = \varphi_2(\vec{r})$ everywhere inside

$\Rightarrow \varphi(\vec{r})$ inside is unique

Conductors (vs. Insulators)

We've devoted some time to computing $\vec{E}$ in various systems, that is the force per unit charge.

Now the question arises: what does a charge actually do in the presence of that force?

For a point charge in free space, we could write $\vec{F} = q\vec{E} = m\vec{a}$ and begin to compute acceleration, etc.

Thought exercise: do charges follow trajectories along field lines?

But what happens when our charge is part of a material? What if we apply $\vec{E}$ to a whole lump of something composed of atoms, nuclei, and electrons? Do these individual, microscopic charges actually respond to $\vec{E}_{external}$ by moving, or are they stuck in place by their interatomic forces (which are, after all, just more electric forces)?

Answer: it depends!

We can reason intuitively that for any given material, the response of the charges in that material must be proportional to the applied $\vec{E}$-field. In fact, we can write:

$$\vec{J} = \sigma\vec{E}$$

$\vec{J}$: current density = amount of charge passing through a unit cross-sectional area per unit time $\frac{[\text{charge}]}{[\text{length}]^2[\text{time}]} = \frac{C}{m^2 s}$

σ: "conductivity" $\left(\frac{C}{m^2 s N} = \frac{C^2 s}{m^3 kg} = \frac{1}{\Omega m}\right)$ "ohm"

$\vec{E}$: electric field inside the material $\left(\frac{N}{C}\right)$

Some typical values for conductivity σ:

	$\sigma\left(\frac{1}{\Omega m}\right)$ ← sometimes written $\mho/m$
silver	6.3×10^{7}
copper	5.9×10^{7}
manganin (alloy)	2.3×10^{6}
silicon	33
salt water	23
drinking water	0.0005 - 0.05
deionized water	5.5×10^{-6}
wood	$10^{-8} - 10^{-11}$
glass	$10^{-10} - 10^{-14}$
sulfur	5×10^{-16}
rubber	$10^{-13} - 10^{-16}$

Conductivity spans 23 orders of magnitude in these materials!
(Compare to other properties of materials, like heat capacity, which differ by a few orders of magnitude.)

What is σ physically? A measure of how tightly e^- are bound to their local nuclei.

metals: e^- are free to "swim around" in an "electron sea"
semiconductors: a few e^- are thermally excited out of their bound, stuck state, to move around
solutions: some charged ions are free to move around
insulators: all e^- are bound to stationary nuclei; no charge can move

Clearly, σ is a property that can take on a continuous range of values. But it turns out that most materials have very high or very low σ. We'll simplify and consider only 2 categories of materials:

conductors: charge flows freely ("near instantaneously") in response to $\vec{E}$: $\sigma \sim \infty$

insulators: no charge flows at all: $\sigma \sim 0$

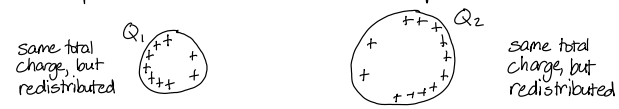
<u>A Tale of 2 Spheres</u>

Consider bringing 2 charged spheres together:

As we bring them close, the like charges will repel each other. If the spheres are insulators, that's the end of the story: the + charges may not like each other but they have to stay put where they started on the surface of each sphere.

But what if the spheres are conductors?
→ 2 competing effects:
① + charges on a single sphere want to get as far away from each other as possible; intuitively they must spread out evenly over sphere

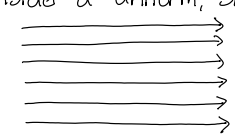
② as the spheres approach each other, the individual + charges on one sphere will repel the + charges on the other, so extra + charges will build up on the far side of each sphere:



(Of course in real solids it's not the + charges that move, it's the negative e^-, but the effect of moving e^- is that the net positive charge redistributes.)

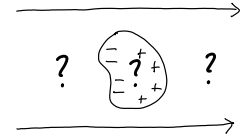
In an idealized conductor, the charges redistribute in response to the applied $\vec{E}$-field near-instantaneously. We're still doing <u>electrostatics</u> (we'll need a lot more tools before we're ready to consider moving charges). So let's just consider the conductor a split second after we've turned on the $\vec{E}$-field, when the charges have already finished redistributing and are stationary again.

Consider a uniform, static $\vec{E}$-field:



Exercise: how would you make such an $\vec{E}$-field?

Now plop down an uncharged conductor into this field:



The charges in the conductor will respond to the $\vec{E}$-field and reconfigure. Furthermore, the $\vec{E}$-field itself will be modified in response to the reconfigured charges!

Exercise: qualitatively, what will the new total $\vec{E}$-field look like?

① $\vec{E} = 0$ inside the conductor

Why? Well, if $\vec{E}$ didn't vanish, the charges would still be moving in response to $\vec{E}$. So just wait another split second until they stop. What if they don't stop? Well, then we've used a static $\vec{E}$ to create a perpetual motion machine → send it to the patent office!

Conclusion: after charges stop moving (near-instantaneously), $\vec{E} = 0$ inside a conductor.

② $\vec{E}$ is perpendicular to surface of conductor (i.e. $\vec{E}_{\parallel} = 0$)

Why? Same reason as ①; if we had a component of $\vec{E}$ parallel to the surface, the charges would still be moving along the surface.

③ φ is constant throughout a conductor, including surface

Why? $\vec{E} = -\vec{\nabla}\varphi$ so if $\vec{E} = 0$ inside then we'd better have φ constant inside. And if $\vec{E}_{\parallel} = 0$ on surface then φ had better not be changing as we run around on the surface.

④ $\rho = 0$ inside a conductor

Why? Because Gauss' law says $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$, so if $\vec{E} = 0$ then $\rho = 0$.

Exercise: Does this seem counter-intuitive? If the charge in the conductor wants to get as far away from itself as possible, then why doesn't it spread throughout instead of concentrating on surface?
(a) Compute the energy of Q spread over surface of sphere, radius R
(b) Compute the energy of Q spread uniformly throughout sphere

⑤ At any point *just* outside the conductor, $\vec{E}_{\perp} = \sigma/\varepsilon_0$, where σ is the local surface charge on the conductor.

Why? To a point *just* outside the surface, it looks like a plane with surface charge σ.

Exercise: Use Gauss' law to prove claim ⑤

⑥ Pressure on the surface of a conductor: $P = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2}E^2$ outward

where σ is the local surface charge on the conductor, and E is the field *just* outside the conductor.

Why? Consider a tiny patch of the surface, with charge σda. Then the force should be $\vec{f} = \vec{E}\sigma da$. But which field do we use, $\vec{E}_{inside} = 0$ or $\vec{E}_{outside} = \sigma/\varepsilon_0\,\hat{n}$ (when $\hat{n} \perp$ surface)?

Answer: we use the average field, $\vec{E}_{avg} = \frac{1}{2}(\vec{E}_{inside} + \vec{E}_{outside}) = \frac{\sigma}{2\varepsilon_0}\hat{n}$

Why $\vec{E}_{avg}$? Total field consists of field produced by charges on the patch itself + field produced by everything else: $\vec{E}_{tot} = \vec{E}_{patch} + \vec{E}_{else}$
But $\vec{E}_{patch}$ can't contribute to the force on the patch.
So the field we need to use should be $\vec{E}_{else}$.

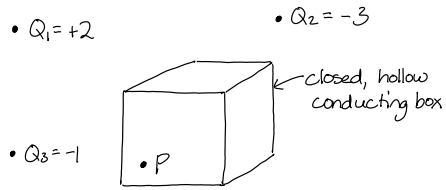
$$\vec{E}_{\substack{total\\outside}} = \vec{E}_{else} + \frac{\sigma}{2\varepsilon_0}\hat{n} \;;\; \vec{E}_{\substack{total\\inside}} = \vec{E}_{else} - \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$\Rightarrow \vec{E}_{else} = \frac{1}{2}\left(\vec{E}_{\substack{total\\inside}} + \vec{E}_{\substack{total\\outside}}\right) = \vec{E}_{\substack{total\\avg}} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

Therefore, $P = \frac{f}{da} = \vec{E}_{avg}\sigma = \frac{\sigma^2}{2\varepsilon_0}$ directed outwards.

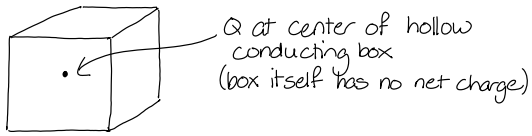
Exercises on conductors:

(a)



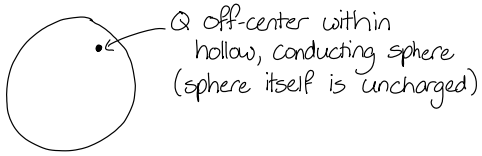
What is $\vec{E}$ at point P, inside the box? Why?

(b)



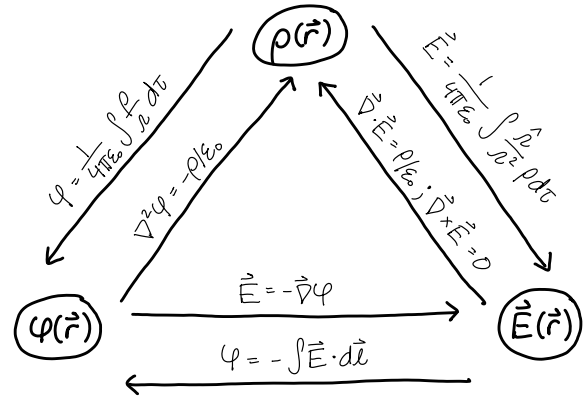
What is $\vec{E}$ inside box & outside box?

(c)



What is $\vec{E}$ outside the sphere?

Summary



Energy:

Collection of point charges: $W = \frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{\mathscr{r}_{ij}}$

Electric field: $W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,d\tau$

Laplace's equation: $\nabla^2\varphi = 0$ (regions with no charge)

1. Average value on surface of sphere = value at center
2. No local maxima or minima
3. Unique solution if φ is specified on boundary

Conductors:

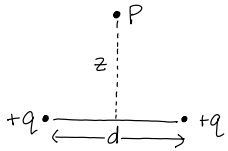
1. $\vec{E} = 0$ inside
2. $\vec{E}_{\parallel} = 0$ on surface
3. φ is constant throughout
4. $\rho = 0$ inside
5. $\vec{E}_{\perp} = \sigma/\varepsilon_0\,\hat{n}$ just outside
6. pressure on surface is $\sigma^2/2\varepsilon_0 = \varepsilon_0 E^2/2$ outwards

Next time:

* Laplace's eqn in more detal
* more applications to conductors

Solutions to Exercises

$\vec{E}$-field at:



$$\varphi(P) = \frac{2q}{4\pi\varepsilon_0\sqrt{\left(\frac{1}{2}d\right)^2 + z^2}}$$

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{d\varphi}{dz}\hat{z} = \frac{2qz\,\hat{z}}{4\pi\varepsilon_0\left(\left(\frac{1}{2}d\right)^2 + z^2\right)^{3/2}}$$

Inconsistency: Both are correct.

(a) is the energy to assemble a system of pt charges, assuming each pt charge already exists

(b) is the total energy, including any energy to assemble pt charges. Note that the energy to assemble a single pt charge diverges. In the real world, our pt charges are so small that we usually average over them & consider only continuous charge distributions, such that the charge at any single point is vanishingly small. This works to describe the classical world but it doesn't get around the awkward fact that the energy of an electron is, in theory, infinite.

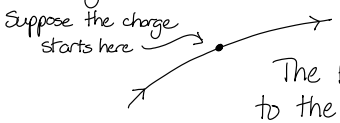
Why is (b) always positive? Any charge distribution at all will produce non-zero fields. Any non-zero field is always a higher energy configuration than $\vec{E}=0$, which we know b/c charges would always like to redistribute themselves to cancel any field.

Where is energy stored?

Classically, can think of it either way.
Eventually (General relativity, etc.) need to consider the energy as stored in the field.

Do charges flow along field lines?

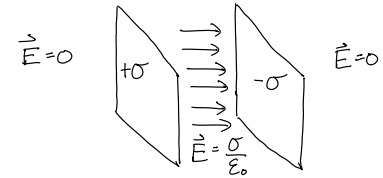
In general, no. Consider a field line with any curvature:



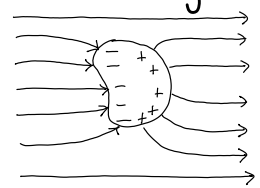
The force on the charge at this point is tangent to the field line. So the acceleration of the charge will be tangent to the field line. But in order to follow a curved trajectory, there would need to be some component of acceleration perpendicular to the trajectory.

How to make a uniform, static $\vec{E}$-field?

2 parallel infinite charged sheets:



$\vec{E}$-field of uncharged conductor in uniform applied $\vec{E}$?



Far from the conductor, $\vec{E}$ is unchanged, but close to it the field lines are bent in perpendicular to the surface.

Energy of charge Q on a sphere of radius R

(a) surface

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \text{ outside} \Rightarrow \underset{\text{energy}}{E} = \frac{\varepsilon_0}{2}\int_R^\infty \underset{\text{field}}{E^2}\,dr = \frac{1}{8\pi\varepsilon_0}\frac{Q^2}{R}$$

(b) volume

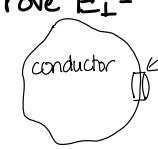
Still have $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ outside, but now $\vec{E} \neq 0$ inside as well

$\Rightarrow E_{tot} = E_{out} + E_{in}$, where E_{out} is what we calculated in (a),

and $\underset{\text{energy}}{E_{in}} = \frac{\varepsilon_0}{2}\int_0^R \underset{\text{field}}{E_{in}^2}\,dr > 0$

The energy of this configuration is higher (in fact, it's $\frac{3}{20\pi\varepsilon_0}\frac{Q^2}{R}$)

Prove $\vec{E}_\perp = \sigma/\varepsilon_0\,\hat{n}$ just outside conductor



Take a small Gaussian box, area A

Gauss' law: $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} q_{enclosed} = \frac{1}{\varepsilon_0}\sigma A$

- inside surface of box: $\vec{E}=0$ inside conductor
- side walls of box: $\vec{E}_{\parallel}=0$ on conductor surface, so there is no component of $\vec{E}$ exiting side walls of box
- outside surface of box: $\vec{E}$ perp to box & uniform if box is small enough

$\Rightarrow \oint_S \vec{E}\cdot d\vec{a} = E_{out}A = \frac{\sigma A}{\varepsilon_0} \Rightarrow E_{out} = \frac{\sigma}{\varepsilon_0}$ perp to box