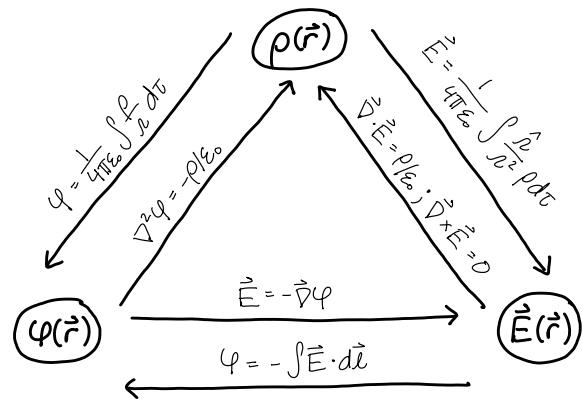


Physics 153 (Hoffman)  
Lecture #4  
Tues, Feb 10, 2009

## Laplace's Equation

Last time:



Energy:

$$\text{Collection of point charges: } W = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}}$$

$$\text{Electric field: } W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Conductors:

- ①  $\vec{E} = 0$  inside
- ②  $\vec{E}_{||} = 0$  on surface
- ③  $\varphi$  is constant throughout
- ④  $\rho = 0$  inside
- ⑤  $\vec{E}_{||} = \sigma/\epsilon_0 \hat{n}$  just outside
- ⑥ pressure on surface is  $\sigma^2/2\epsilon_0 = \epsilon_0 E^2/2$  outwards

Laplace's equation:  $\nabla^2 \varphi = 0$  (regions with no charge)

- ① Average value on surface of sphere = value at center
- ② No local maxima or minima
- ③ Unique solution if  $\varphi$  is specified on boundary

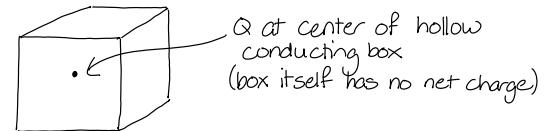
①

### Exercises on conductors:

- (a)
- $Q_1 = +2$
  - $Q_2 = -3$
  - $Q_3 = -1$
- 

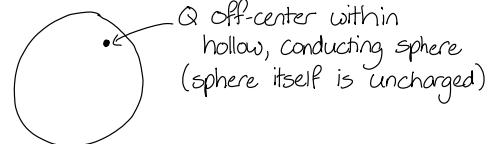
What is  $\vec{E}$  at point P, inside the box? Why?

(b)



What can you say about  $\vec{E}$  inside box & outside box?

(c)



What is  $\vec{E}$  outside the sphere?

②

(3)

## Goals for today:

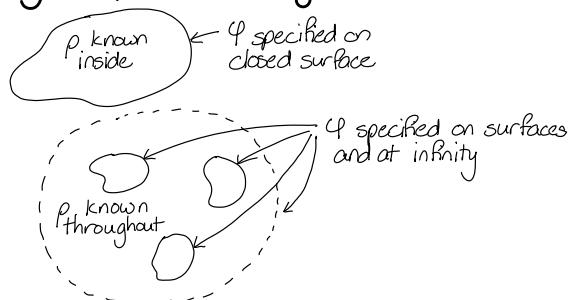
- \* Laplace's eqn: uniqueness of solutions (given appropriate boundary conditions)
  - \* Capacitors
  - \* Image charges
  - \* Methods for solving Laplace's equation
    - relaxation method
    - separation of variables
      - example in Cartesian coordinates
      - Fourier transforms
- (Next time: spherical coordinates, Legendre polynomials, etc.)

## Laplace's equation: Uniqueness theorems

There is a unique solution for  $\varphi$  given appropriate boundary conditions.

### ① $\varphi$ specified on boundary, $\rho$ specified throughout volume

Note: we can think of a simple boundary:



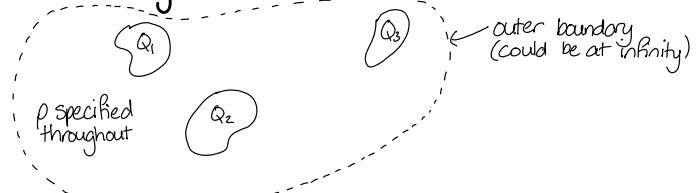
Proof: Suppose there are 2 functions  $\varphi_1(\vec{r})$  and  $\varphi_2(\vec{r})$  which take the same values on all surfaces, and which both satisfy Poisson's eqn:

$$\nabla^2 \varphi_1 = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 \varphi_2 = -\frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 (\varphi_1 - \varphi_2) = 0 \text{ inside}$$

But  $\varphi_1 - \varphi_2 = 0$  everywhere on surfaces ( $\varphi_1$  and  $\varphi_2$  have same values) and  $\varphi_1 - \varphi_2$  satisfies Laplace's eqn inside  
 $\Rightarrow \varphi_1 - \varphi_2$  can have no maxima or minima inside  
 $\Rightarrow \varphi_1 - \varphi_2 = 0$  everywhere inside  
 $\Rightarrow \varphi_1(\vec{r}) = \varphi_2(\vec{r})$  everywhere inside  
 $\Rightarrow \varphi(\vec{r})$  inside is unique

(4)

### ② volume bounded by conductors, each with known total charge $Q_i$ ; $\rho$ known throughout volume



Proof: suppose there are two fields  $\vec{E}_1$  and  $\vec{E}_2$ , each satisfying:

$$\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0} \quad (\text{throughout volume})$$

and  $\oint_{\text{ith surface}} \vec{E}_1 \cdot d\vec{a} = \frac{Q_i}{\epsilon_0}$  and  $\oint_{\text{ith surface}} \vec{E}_2 \cdot d\vec{a} = \frac{Q_i}{\epsilon_0}$  (on surfaces immediately surrounding each conductor)

and  $\oint_{\text{outer boundary}} \vec{E}_1 \cdot d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}$  and  $\oint_{\text{outer boundary}} \vec{E}_2 \cdot d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}$  (outer boundary, possibly at infinity)

The difference  $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$  obeys  $\vec{\nabla} \cdot \vec{E}_3 = 0$  throughout volume and  $\oint_{\text{ith surface}} \vec{E}_3 \cdot d\vec{a} = 0$  on each boundary surface.

Now, a trick:  $\vec{\nabla} \cdot (\varphi_3 \vec{E}_3) = \varphi_3 (\vec{\nabla} \cdot \vec{E}_3) + \vec{E}_3 \cdot (\vec{\nabla} \varphi_3) = -(\vec{E}_3)^2$

$$\int_V \vec{\nabla} \cdot (\varphi_3 \vec{E}_3) dV = - \int_V (\vec{E}_3)^2 dV$$

↓ divergence theorem

$$\oint_{\text{all surfaces}} \varphi_3 \vec{E}_3 \cdot d\vec{a} = \sum_i \oint_{\text{ith surface}} \varphi_3 \vec{E}_3 \cdot d\vec{a}$$

But  $\varphi_3$  and  $\vec{E}_3$  must each be constant on each surface (because surfaces are conducting)  $\Rightarrow \varphi_3 = \varphi_1 - \varphi_2$  is constant on each surface.

$$\Rightarrow \int_V (\vec{E}_3)^2 dV = - \sum_i \varphi_3 \oint_{\text{ith surface}} \vec{E}_3 \cdot d\vec{a} = 0$$

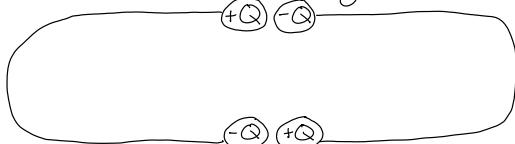
$$\Rightarrow \vec{E}_3 = 0 \text{ everywhere}$$

(5)

**Exercise:** Consider a configuration of 4 charged conductors:



Now add 2 thin conducting wires:



Will the charges stay put?

### Capacitance:

Consider 2 conductors, with charge  $+Q$  on one and  $-Q$  on the other.  $\varphi$  is constant over each one, so we can unambiguously define

$$\varphi = \varphi_+ - \varphi_- = \int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$$

Without knowing or bothering to calculate exactly how each  $Q$  distributes over each conductor, we can say that  $\vec{E} \propto Q$  because

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \hat{r} dr \text{ so doubling } \rho \text{ doubles } \vec{E}$$

(And we know that the whole charge distribution  $\rho(\vec{r})$  just scales with the total charge, because all equations are linear, so clearly having  $\rho(\vec{r})$  scale with the total charge is one solution. By the uniqueness theorems, it must be the solution.)

OK, if  $\vec{E} \propto Q$  then  $\varphi \propto Q$  so we can define a constant  $C$ :

$$\text{capacitance } C = \frac{Q}{\varphi} \quad (\text{units = Farads = Coulomb/Volt})$$

\* Note: if you ever hear anybody talking about the capacitance of a single conductor, just assume the second conductor is at infinity

\* Note: to find potential difference  $\varphi$ , always do line integral from (-) to (+) so that  $\varphi$  will come out positive, and  $C$  will be positive (by definition)

(6)

### Energy of a capacitor:

To charge up a capacitor, you have to do work to move charge:

$$dW = \varphi dq$$

↑ increment of charge you are moving

potential difference between conductors

due to charge that has already been moved =  $\frac{q}{C}$

So to charge from  $q=0$  to  $q=Q$  requires total work:

$$W = \int_0^Q \left( \frac{q}{C} \right) dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \varphi^2$$

**Example:** What is the capacitance of two parallel conducting plates, each of area  $A = 1\text{cm}^2$  separated by distance  $d = 1\text{mm}$ ?



If we put  $+Q$  on top and  $-Q$  on bottom, they will each spread nearly uniformly across each plate (assuming  $A$  large,  $d$  small so we can neglect edge effects).

$$\Rightarrow \text{charge density } \sigma = Q/A$$

$$\Rightarrow \text{each plate contributes } \vec{E} = \sigma/\epsilon_0 \hat{n} \text{ to the region between the plates}$$

$$\Rightarrow \vec{E}_{\text{tot}} = Q/\epsilon_0 A \hat{n} \text{ between plates}$$

$$\Rightarrow \varphi = - \int \vec{E}_{\text{tot}} \cdot d\vec{l} = \frac{Qd}{\epsilon_0 A}$$

$$\Rightarrow C = \frac{Q}{\varphi} = \frac{\epsilon_0 A}{d} = \frac{(8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2)(10^{-4} \text{ m})}{10^{-3} \text{ m}} \approx 9 \times 10^{-13} \text{ F}$$

**Exercise:** What is the capacitance of 2 concentric spherical metal shells with radii  $a$  (inner) and  $b$  (outer)?

(7)

## Method of Images

We've now talked a bunch about charges redistributing on conductors. We've proved that they do so uniquely, and we've used that idea to define a useful device for storing a known quantity of electrical energy.

But how do we calculate exactly how charges redistribute?

Here is one trick, which is admittedly useful only in cases of high symmetry. Consider a single point charge, held a distance  $d$  above an infinite conducting plane:



Some negative charge must distribute itself somehow on the plane in the region under the conductor.

We have boundary conditions:

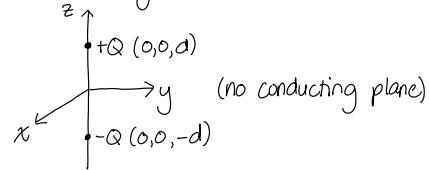
$$\textcircled{1} \quad \varphi = 0 \text{ for } z = 0$$

$$\textcircled{2} \quad \varphi \rightarrow 0 \text{ far from the charge, for } x^2 + y^2 + z^2 \gg d^2$$

And we know  $\rho$  everywhere for  $z > 0$ .

$\Rightarrow$  there must be a unique solution for  $\varphi$  (in  $z > 0$  region).

Trick: consider a different charge distribution (identical for  $z > 0$ )



$$\varphi(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left[ \underbrace{\frac{1}{(x^2 + y^2 + (z-d)^2)^{1/2}}}_{\text{top charge } (+Q)} - \underbrace{\frac{1}{(x^2 + y^2 + (z+d)^2)^{1/2}}}_{\text{bottom charge } (-Q)} \right]$$

Notice that for this potential:

$$\textcircled{1} \quad \varphi = 0 \text{ for } z = 0$$

$$\textcircled{2} \quad \varphi \rightarrow 0 \text{ for } x^2 + y^2 + z^2 \gg d^2$$

and  $\rho$  is identical to the original  $\rho$  for  $z > 0$ .

Uniqueness theorem: this "fake"  $\varphi(x, y, z)$  is our actual  $\varphi(x, y, z)$ !

(8)

Surface charge on the conducting plane at  $z = 0$ :

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z} \Big|_{z=0} = \frac{-Qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

[Negative, as expected, and maximized at  $(x, y) = (0, 0)$ ]

$$Q_{\text{surf total}} = \int_0^\infty dr \int_0^{2\pi} r d\phi \frac{-Qd}{2\pi(r^2 + d^2)^{3/2}} = -Q$$

Force exerted by conductor on charge:

$Q$  must be attracted to plane (i.e.  $-\hat{z}$  direction)

Since  $\varphi$  is the same for real problem and "fake" problem for  $z > 0$ ,

$E$  must be the same too, so  $F$  is the same:

$$\vec{F} = -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{(2d)^2} \hat{z}$$

**Exercise:** What is the energy of the charge configuration (point charge & conducting plane)?

**Challenge:** A point charge  $Q$  is situated a distance  $a$  from the center of a grounded conducting sphere of radius  $R$ . What is the potential outside the sphere?

Ok, that's nice for cute, high-symmetry problems, but what about in general? How do we know how charges redistribute in general?

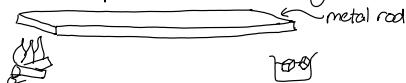
⑨

### Methods for solving Laplace's equation: $\nabla^2\varphi=0$

#### ⑩ Relaxation method

Physical intuition: Laplace's equation actually describes many different physical systems, not just electrostatic potential.

For example, temperature of a system in thermal equilibrium:



There is a temperature gradient across the rod,  $\vec{\nabla}T$ . Energy is transported (in the form of heat) from high T to low T according to:

$$\underbrace{\vec{P}}_{\substack{\text{power} \\ \text{cross-} \\ \text{sectional} \\ \text{area}}} = -K \vec{\nabla}T \quad \text{thermal conductivity}$$

Now we can take the divergence of both sides:

$$\vec{\nabla} \cdot \underbrace{\frac{\vec{P}}{A}}_{\text{a}} = -K \nabla^2 T$$

all along the rod except for the endpoints, this vanishes, because there are no energy sources or sinks (energy is conserved)

If we know the values at the boundary, say  $T_1=500^\circ\text{C}$  and  $T_2=0^\circ\text{C}$ , where there are sources and sinks of heat that fix the temperature, then we can solve Laplace's equation to find  $T$  everywhere else in the rod.

We can see from physical intuition that the solutions to Laplace's equation are very smooth: they must average somehow over boundary conditions.

Mathematically, we already know that the value at the center of a sphere is the average of the values over the sphere, so there are no maxima or minima.

#### ⑪ Numerical method for solving Laplace's equation

Example: Suppose we have a square cross-section coaxial cable, and we apply  $\varphi=100\text{V}$  to the center conductor, but ground the outside,  $\varphi=0$ . What is the potential distribution throughout the interior?

Solution:

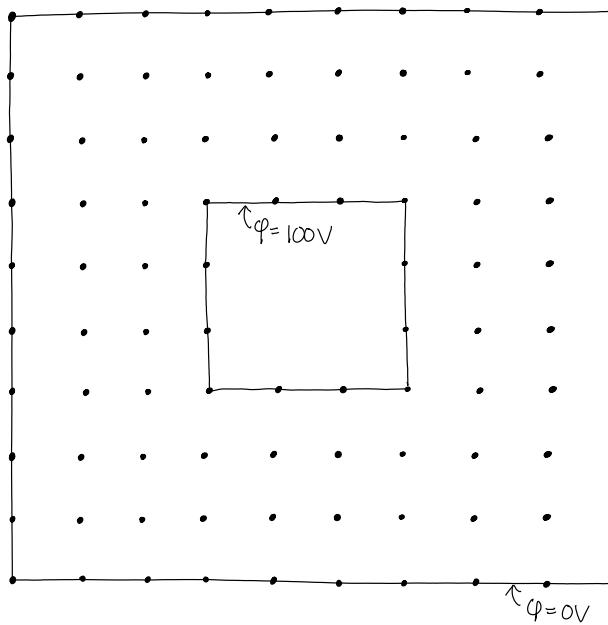
① Set up a grid

② Assign some reasonable starting values (guess)

③ Start systematically averaging 4 neighbors to get center value

④ When no values change by more than some error value (say 1%) on a given pass, then you're done.

**Tedious exercise:** in case you know everything already and you're really bored, you can entertain yourself by solving this problem.



⑪

## ② Separation of variables

Strategy: look for solutions which are products of functions, each of which depends on only one of the coordinates.

Cartesian coordinates:

$$\nabla^2 \varphi = \frac{\partial^2}{\partial x^2} \varphi + \frac{\partial^2}{\partial y^2} \varphi + \frac{\partial^2}{\partial z^2} \varphi = 0$$

Now suppose that  $\varphi(x, y, z) = \underbrace{X(x)Y(y)Z(z)}_{\text{product of 3 functions}} \text{ each depends on only single variable}$

Apply Laplacian operator  $\nabla^2$ :

$$Y(y)Z(z) \frac{\partial^2 X}{\partial x^2} + X(x)Z(z) \frac{\partial^2 Y}{\partial y^2} + X(x)Y(y) \frac{\partial^2 Z}{\partial z^2} = 0$$

$$\Rightarrow \frac{\frac{\partial^2 X}{\partial x^2}}{X(x)} + \frac{\frac{\partial^2 Y}{\partial y^2}}{Y(y)} + \frac{\frac{\partial^2 Z}{\partial z^2}}{Z(z)} = 0$$

This is now the sum of 3 functions, each of which depends only on a single variable. Each variable,  $x, y, z$ , can vary independently (we can move in the  $x$ -direction,  $y$ -direction, or  $z$ -direction) but the sum is still constant.

⇒ Each of the 3 terms must be independently constant.

$$\frac{\partial^2 X}{\partial x^2} = C_1 X(x); \quad \frac{\partial^2 Y}{\partial y^2} = C_2 Y(y); \quad \frac{\partial^2 Z}{\partial z^2} = C_3 Z(z)$$

where  $C_1 + C_2 + C_3 = 0$ .

Now we have 3 separate ordinary differential eqns which we can solve independently. → Much easier!

But wait! What if the real solution to  $\nabla^2 \varphi = 0$  in a particular physical situation wasn't a product  $X(x)Y(y)Z(z)$ ? What if it was  $\varphi(x, y, z) = 3x + 4y + 5z$ ? Haven't we placed too much restriction on our proposed solution to ever have any hope of meeting arbitrary boundary conditions?

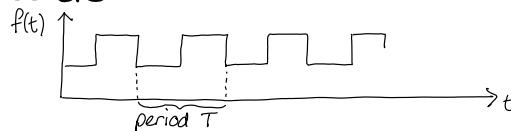
No! Each of those functions  $X(x)$ , etc., actually has an infinite number of solutions (e.g. any cosine will do, if  $C_i < 0$ ). Better yet, all of those solutions are orthogonal, and complete for each dimension.

So by taking a judicious linear combination, we can match any boundary conditions.

⑫

Fourier transforms:

① Discrete:



The discrete (but infinite) set of functions  $\cos\left(\frac{2\pi nt}{T}\right)$  and  $\sin\left(\frac{2\pi nt}{T}\right)$  form a complete, orthogonal basis for the set of all periodic functions  $f(t)$  with period  $T$ .

⇒ We can write any periodic  $f(t)$  with period  $T$  as:

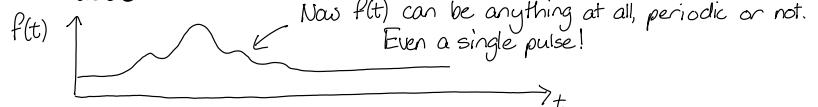
$$f(t) = a_0 + \sum_{n=1}^{\infty} \{ a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \}$$

constant offset

where  $a_n$  and  $b_n$  come from the "inner product" of  $f(t)$  with the basis state, e.g.

$$a_n = \frac{1}{T} \int_0^T f(t) \cos\left(\frac{2\pi nt}{T}\right) dt$$

② Continuous:



The continuous, infinite set of functions  $e^{i\omega t}$  (where  $\omega$  runs over all real numbers from  $-\infty$  to  $+\infty$ ) form a complete, orthogonal basis for the set of all functions  $f(t)$ .

well... all well-behaved functions, e.g. finite # of finite discontinuities, etc

$$f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega t} d\omega$$

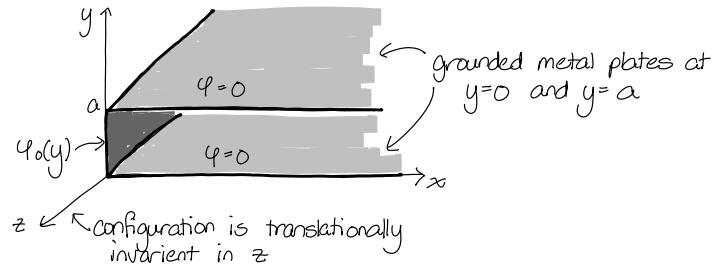
basis vectors  
coefficient of linear combo of basis vectors  
need to integrate instead of add to find linear combo,  
because  $\omega$  is continuous

Now  $\tilde{F}(\omega)$  comes from an "inner product" of  $f(t)$  with basis state:

$$\tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Inner product is a measure of the "overlap" of 2 functions:  $\int_{-\infty}^{\infty} f(t) g^*(t) dt$

Example:



Find the potential inside this "slot", for  $x > 0$ ,  $0 < y < a$ , and all  $z$ . First note that it is  $z$ -independent, so it reduces to

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

The boundary conditions are:

①  $\varphi = 0$  when  $y = 0$

②  $\varphi = 0$  when  $y = a$

③  $\varphi = \varphi_0(y)$  when  $x = 0$

(some arbitrary function of  $y$   
that we're going to have to match)

④  $\varphi \rightarrow 0$  as  $x \rightarrow \infty$

Look for solutions of the form  $\varphi(x, y) = X(x)Y(y)$

They must satisfy

$$\frac{d^2X(x)}{dx^2} = C_1 X(x) \quad \text{and} \quad \frac{d^2Y(y)}{dy^2} = C_2 Y(y) \quad \text{where } C_1 + C_2 = 0$$

So one of  $C_1, C_2$  must be positive and the other negative.

Let's choose  $C_1 = k^2$  and  $C_2 = -k^2$ .

$$\frac{d^2X}{dx^2} = k^2 X \Rightarrow X(x) = A e^{kx} + B e^{-kx}$$

$$\frac{d^2Y}{dy^2} = -k^2 Y \Rightarrow Y(y) = C \sin(ky) + D \cos(ky)$$

$$\varphi(x, y) = [A e^{kx} + B e^{-kx}] [C \sin(ky) + D \cos(ky)]$$

We reject this b/c it would cause  $\varphi(x, y)$  to blow up as  $x \rightarrow \infty$ .

We reject this because we need  $\varphi(x, y) = 0$  when  $y = 0$ .

$$\varphi(x, y) = C e^{-kx} \sin(ky)$$

$k$  is quantized by the condition  $\sin(ky) = 0$  when  $y = a$   
 $\Rightarrow k = \frac{n\pi}{a}$  for any integer  $n$

⑬

Now we have an infinite set of solutions  $\varphi_n(x, y) = C_n e^{-n\pi x/a} \sin\left(\frac{n\pi}{a} y\right)$  which each meet boundary conditions ①, ② and ④.

But for some arbitrary  $\varphi_0(y)$  in boundary condition ③, none of the  $\varphi_n(x, y)$ 's are going to individually satisfy it. Luckily for us, the  $\varphi_n(x, y)$ 's are complete and orthogonal

hard to prove  
(Dirichlet's theorem)

easy to prove:  
just check inner product  
for  $\varphi_n$  and  $\varphi_m$

So we construct a general solution

$$\varphi(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi}{a} y\right)$$

and choose the coefficients  $C_n$  to satisfy boundary condition ③:

$$\varphi(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = \varphi_0(y)$$

To compute  $C_n$ 's, use Fourier's trick:

$$C_n = \frac{2}{a} \int_0^a \varphi_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

normalization constant  
inner product: Conceptually, measures the "overlap" of function  $\varphi_0(y)$  with the basis function  $\sin\left(\frac{n\pi y}{a}\right)$  over the interval  $(0 \rightarrow a)$  of interest. So it tells us how much of  $\sin\left(\frac{n\pi y}{a}\right)$  we're going to need to put in our sum to make our function  $\varphi_0(y)$ .

Exercise: concrete example of  $\varphi_0(y) = \varphi_0 = \text{constant}$ , solve for  $C_n$ 's

### Summary:

①  $\varphi$  and  $\psi$  are uniquely defined when:

- (a)  $\varphi$  specified on boundary,  $\rho$  specified throughout volume
- (b) volume bounded by conductors, each with known total charge  $Q_i$ ;  $\rho$  known throughout volume

② Capacitors:  $C = \frac{Q}{V}$ ; energy =  $\frac{1}{2} CV^2$

③ Methods for figuring out what's going on in the presence of conductors

- (a) method of images (works only in high symmetry cases)

- (b) numerical relaxation method for Laplace's equation

- (c) separation of variables

④ Fourier transform: express a function as a linear combination of basis functions

(15)

### Exercise Solutions:

**Conductors (a):** We know  $\varphi$  is constant on the surface of the box, because it is a conductor.

$\Rightarrow \varphi_{\text{inside}} = \varphi_{\text{surface}} = \text{constant}$  Why? 2 different explanations:

①  $\varphi$  can't have a local min or max inside box

② clearly  $\nabla \varphi = \text{constant}$  is one solution to  $\nabla^2 \varphi = 0$

so it must be the only solution

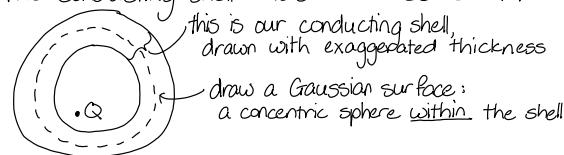
$$\Rightarrow \vec{E} = -\vec{\nabla} \varphi = 0 \text{ inside}$$

**Conductors (b):**  $\vec{E} = 0$  inside or outside (apply Gauss' law in integral form).

It's not easy to calculate the exact form of  $\vec{E}$ , because there isn't spherical symmetry so we don't know (without a lot of work) how the charges will redistribute on the conducting cube.

**Conductors (c):**  $\vec{E} = \frac{Q}{r^2} \hat{r}$  outside, as if  $Q$  were at the center.

Why? The conducting shell must have some thickness:



$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{\text{closed}}$$

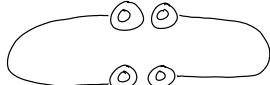
but  $\vec{E} = 0$  because it's within the conductor

$$\Rightarrow q_{\text{closed}} = 0$$

$\Rightarrow -Q$  must have distributed itself on the inside of the shell, leaving  $+Q$  to distribute itself on the outside.

But all that the outside  $+Q$  knows is that the field immediately adjacent to it (just within the shell) is  $\vec{E} = 0$ , so  $+Q$  will distribute itself evenly.

**Conductors (d):** The charges will flow until the final configuration is neutral.



Why? Because clearly this is a solution, if we start this way with neutral conductors; therefore it must be the only solution.

(16)

### Capacitance of spheres:

Put  $+Q$  on the inner sphere;  $-Q$  on the outer one.

$$\text{Between the spheres, } \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

$$\varphi = - \int_b^a \vec{E} \cdot d\vec{l} = \frac{-Q}{4\pi\epsilon_0} \int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0} \left. \frac{1}{r} \right|_b^a = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right)$$

$$C = \frac{Q}{\varphi} = 4\pi\epsilon_0 \frac{ab}{b-a}$$

### Energy of charge & plane:

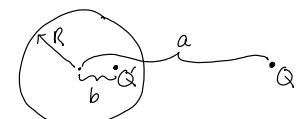
For the "fake" problem with 2 charges, the energy is

$$E = -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{2d}$$

But for a single charge & conducting plane the energy is half this, because we would integrate  $\frac{1}{2} \int E^2 d\tau$  only over upper half-space

**Challenge:** Place an image charge  $Q' = \frac{R}{a} Q$

at a distance  $b = \frac{R^2}{a}$  from the sphere.



See details in Griffiths, page 125.

### Concrete example of Separation of Variables

$$\varphi_o(y) = \varphi_o = \text{constant}$$

$$C_n = \frac{2}{a} \int_0^a \varphi_o \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2\varphi_o}{a} \cdot \frac{a}{n\pi} \left[ -\cos\left(\frac{n\pi y}{a}\right) \right]_0^a = \frac{2\varphi_o}{n\pi} [-\cos(n\pi) + 1]$$

$$C_n = \begin{cases} 0, & n \text{ even} \\ \frac{4\varphi_o}{n\pi}, & n \text{ odd} \end{cases}$$

$$\Rightarrow \varphi(x,y) = \frac{4\varphi_o}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$