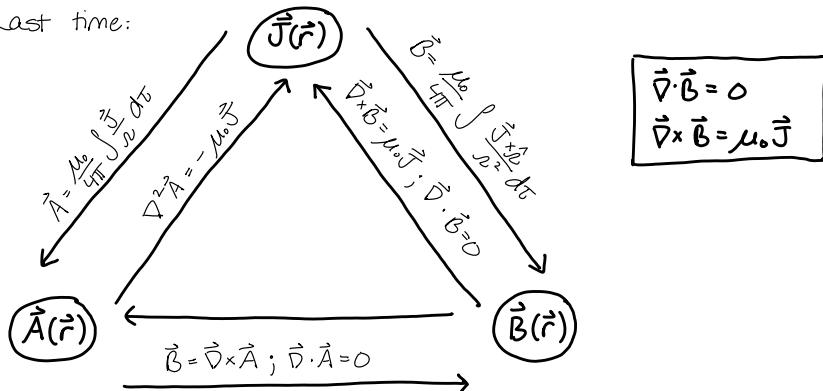
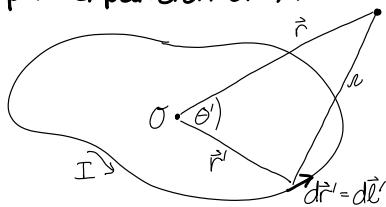


Magnetization

Last time:



Multipole expansion of \vec{A}



$$\frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos \theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \theta')$$

$$\begin{aligned} A(\vec{r}) &= \frac{\mu_0}{4\pi} \int \frac{\vec{I}(\vec{r}')}{r} dr' = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r} d\vec{l}' \\ &= \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \theta') d\vec{l}' \\ &= \frac{\mu_0 I}{4\pi} \left\{ \underbrace{\frac{1}{r} \oint d\vec{l}'}_{=0 \text{ (closed loop)}} + \frac{1}{r^2} \oint r' \cos \theta' d\vec{l}' + \frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) d\vec{l}' + \dots \right\} \end{aligned}$$

$$A_{\text{dip}}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \theta' d\vec{l}' = \frac{\mu_0 I}{4\pi r^2} \left(-\hat{r} \times \oint d\vec{a}' \right)$$

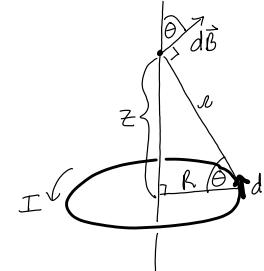
math trick area of loop

①

Define: **magnetic dipole** $\vec{m} = I \oint d\vec{a} \cdot \vec{I} \hat{a}$
where \hat{a} = area of loop, vector pointed normal to area
 $\Rightarrow A_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

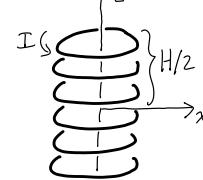
Example: Why I've been at Harvard 4 years and only just moved into a lab last month.

① What is the \vec{B} -field on-axis of a current loop?



$$\begin{aligned} dB_z &= dB \cos \theta = \frac{\mu_0}{4\pi} \left| \frac{\vec{I} \times \hat{r}}{r^2} \right| d\vec{l}' \cos \theta = \frac{\mu_0 I}{4\pi} \frac{\cos \theta}{z^2 + R^2} d\vec{l}' \\ \vec{B} &= \hat{z} \oint dB_z = 2\pi R \oint dB_z \hat{z} = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \hat{z} \end{aligned}$$

② What is the \vec{B} -field at the center of a solenoid with N turns, over total height H ?



$$\begin{aligned} \vec{B} &= \hat{z} \int_{-H/2}^{H/2} \frac{\mu_0 (IN/H) dz}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} = \frac{\mu_0 IN R^2}{2H} \int_{-H/2}^{H/2} \frac{dz}{(z^2 + R^2)^{3/2}} \hat{z} \\ &= \frac{\mu_0 IN}{2H} \frac{z \hat{z}}{(z^2 + R^2)^{1/2}} \Big|_{-H/2}^{H/2} = \frac{\mu_0 IN}{2} \frac{1}{(H^2 + R^2)^{1/2}} \hat{z} \end{aligned}$$

②

(3)

③ What is the dipole moment of the solenoid?

$$\vec{m} = NIa\hat{z}$$

$$= \pi R^2 NI \hat{z}$$

④ What is the vector potential of the solenoid?

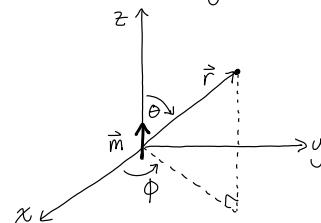
$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \quad (\vec{r} = \text{distance from center of solenoid})$$

⑤ Suppose we have a 9 Tesla solenoid, with $I = 80$ Amps, diameter 5 cm, and height 30 cm. How many turns?

$$N = \frac{2B(H^2 + R^2)^{1/2}}{\mu_0 I} = \frac{2(0.025^2 + 0.3^2)^{1/2}}{(4\pi \times 10^{-7})(80)} = 53,900 \text{ turns}$$

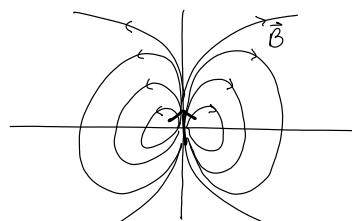
⑥ What is the stray field 10 m away from the solenoid?

Put a dipole at the origin, compute \vec{B} :



$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$$

$$\begin{aligned} \vec{B}_{dip}(\vec{r}) &= \vec{\nabla} \times \vec{A} = \frac{\mu_0 M}{4\pi} \left\{ \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_\phi) \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \hat{\theta} \right\} \\ &= \frac{\mu_0 M}{4\pi} \left\{ \frac{1}{r^3 \sin\theta} \frac{\partial}{\partial\theta} (\sin^2\theta) \hat{r} - \frac{\sin\theta}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \hat{\theta} \right\} \\ &= \frac{\mu_0 M}{4\pi} \left\{ \frac{2 \cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^2} \hat{\theta} \right\} \end{aligned}$$



(4)

$$M = \pi R^2 NI = \pi (0.025)^2 (53,900) (80) = 8470 \text{ A} \cdot \text{m}^2$$

On axis: $\theta = 0$

$$\begin{aligned} \vec{B} &= \frac{\mu_0 M}{4\pi} \cdot \frac{2}{z^3} \hat{z} = \frac{(4\pi \times 10^{-7})(8470)}{4\pi} \frac{2}{(10)^3} \hat{z} = 1.69 \times 10^{-6} \text{ Tesla} \\ &= 17 \times 10^{-3} \text{ Gauss} = 17 \text{ mG} \end{aligned}$$

Off axis: $\theta = \pi/2$

$$\vec{B} = \frac{\mu_0 M}{4\pi} \frac{1}{r^3} (-\hat{z}) = 8.5 \text{ mG}$$

⑦ How much does this deflect a 5 keV electron over 1 m? (assuming \vec{B} is \perp to \vec{v} of electron)

$$5 \text{ keV} = 5000 \text{ eV} = 5000 (1.6 \times 10^{-19} \text{ Coulomb})(1 \text{ Volt}) = 8 \times 10^{-16} \text{ Joules}$$

$$\begin{aligned} &= \frac{1}{2} m_e v^2 \\ &\Rightarrow v = \sqrt{\frac{2(8 \times 10^{-16} \text{ J})}{9 \times 10^{-31} \text{ kg}}} = 4.2 \times 10^7 \text{ m/s} \end{aligned}$$

(ok, this is \gg to c so to be more exact we should take special relativity into account, but it's a decent approximation)

$$\Delta t = \frac{1 \text{ m}}{4.2 \times 10^7 \text{ m/s}} = 2.4 \times 10^{-9} \text{ s}$$

$$F = qvB = (1.6 \times 10^{-19} \text{ C})(4.2 \times 10^7 \text{ m/s})(1.69 \times 10^{-6} \text{ T}) = 1.13 \times 10^{-17} \text{ N}$$

$$\Delta x = \frac{1}{2} a(\Delta t)^2 = \frac{1}{2} \frac{F}{m_e} (\Delta t)^2 = \frac{1}{2} \frac{1.26 \times 10^{-18} \text{ N}}{9 \times 10^{-31} \text{ kg}} (2.4 \times 10^{-9} \text{ s})^2 = 3.6 \times 10^{-3} \text{ m}$$

If we're trying to use the final position of that electron to image a material with atomic resolution, then 3.6 mm of deflection is unacceptable.

What do we do?

Idea #1: magnetic shielding

→ how do magnetic fields behave in materials?

(5)

Magnetic materials: materials in which atoms (or just electrons) have magnetic dipole moments \vec{m} .

$$\vec{M} = \text{magnetization} = \vec{m}/\text{volume}$$

How do individual \vec{m} arise?

First we need to know how \vec{m} behaves in \vec{B} :

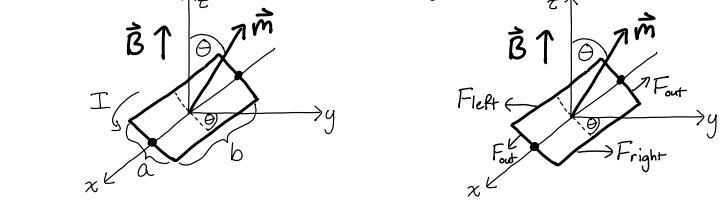
- torque?
- force?
- energy?

Torque on a dipole:

Note that any current loop can be considered as a superposition of rectangular current loops.



What is the torque on a rectangular current loop?



$$F_{\text{left}} = F_{\text{right}} = IbB$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \alpha/2 F \sin\theta \hat{x}$$

$$\vec{\tau} = IabBs \sin\theta \hat{x} = mB \sin\theta \hat{x} \Rightarrow \vec{\tau} = \vec{m} \times \vec{B}$$

Energy of a dipole:

$$dU = -\tau d\theta = -mBs \sin\theta d\theta$$

$$U = -mB \cos\theta$$

(Sanity check on sign: \vec{m} wants to align with \vec{B} , so when $\theta=0$, energy should be minimum.)

$$U = -\vec{m} \cdot \vec{B}$$

(6)

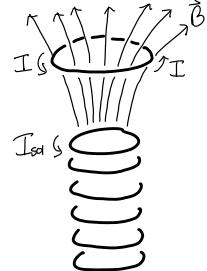
Force on a dipole:

In a uniform field, the net force on a current loop is zero:

$$\vec{F} = I \oint (d\vec{l} \times \vec{B}) = I(\oint d\vec{l}) \times \vec{B} = 0$$

But in a non-uniform \vec{B} , we can't take it outside the integral.

Consider a dipole in the fringe field of a solenoid:



\vec{B} from solenoid is spreading out.
Flux lines can't start or stop (no magnetic monopoles) \rightarrow must be continuous
 \rightarrow must have radially outward component.
 \rightarrow So \vec{F} on upper loop must have downward component
 $\rightarrow \vec{m}$ is attracted to stronger field

In fact, $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ [will derive on homework]

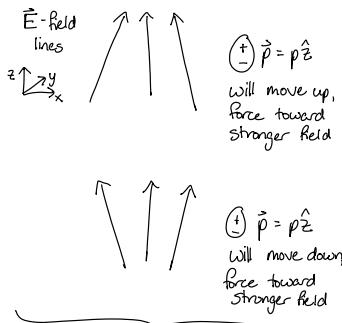
Note: \vec{F} on electric dipole = $\vec{p} \cdot \vec{\nabla} \vec{E}$
which happens to be = $\vec{\nabla}(\vec{p} \cdot \vec{E})$

But $\vec{\nabla}(\vec{m} \cdot \vec{B}) \neq (\vec{m} \cdot \vec{\nabla}) \vec{B}$

(7)

Reminder: Electric fields in matter

materials always polarize along the direction of \vec{E}
i.e. \vec{P} is ALWAYS parallel to \vec{E} , even if its magnitude is nonlinear



This case will prevail for ALL real dielectrics, because \vec{P} will always point in same direction as \vec{E}

Magnetic fields in matter:

We will find empirically that some materials (e.g. bismuth) move towards weaker field, and some materials (e.g. MnSO₄) move towards stronger field.

Paramagnets: analogous to polar materials

have unpaired electron spins that act like magnetic dipoles
applied magnetic field rotates these dipoles
but not complete rotation b/c of thermal energy

$$\Rightarrow \vec{M} = \frac{Nm^2}{k_B T} \vec{B}$$

where m is built-in, microscopic moment
and N is # of m 's per volume

(8)

Diamagnets: analogous to non-polar materials

magnetic field induces a magnetic dipole

(the effect is very weak, but it's present in ALL materials, unlike paramagnetism which is present only in materials with unpaired spins).

Lens' law: will cover in next chapter

idea: electromotive force opposes changing \vec{B} -field

→ induced dipole \vec{m} is opposite applied \vec{B} !

⇒ these materials will be repelled by \vec{B} -field

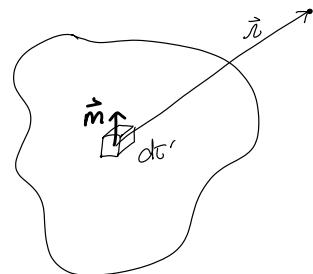
Note: diamagnetic effect is always present

but when paramagnetic effect is also present, it dominates.

(9)

Field of a magnetized object

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{n}}{r^2}$$



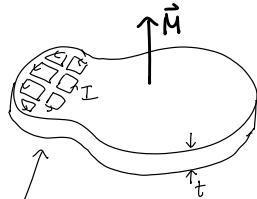
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(r') \times \hat{n}}{r'^2} dV'$$

↓ math

$$\vec{A}(\vec{r}') = \frac{\mu_0}{4\pi} \int \underbrace{\frac{1}{r'} [\vec{v}' \times \vec{M}(r')]}_{\text{volume bound current}} dV' + \frac{\mu_0}{4\pi} \oint \underbrace{\frac{1}{r'} [\vec{M}(r') \times d\vec{a}']}_{\text{surface bound current}}$$

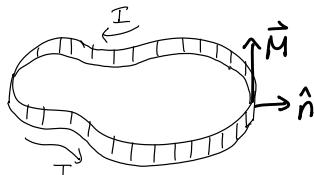
$$\vec{K}_b = \vec{\nabla} \times \vec{M}$$

Surface current:



uniform \vec{M}
⇒ all internal currents vanish

Each tiny loop has thickness t and area a , so in terms of \vec{M} the dipole moment of a single loop is $m = Mat$. But in terms of I , $m = Ia$, so we must have $I = Mt$.
 $\Rightarrow K_b = I/t = M$



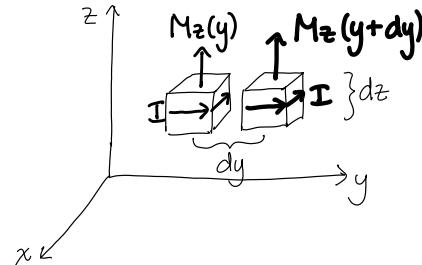
In terms of the directions of \vec{M} and \hat{n} at the surface, we can write:

$$\vec{K}_b = \vec{M} \times \hat{n}$$

(10)

Volume bound current:

When \vec{M} is non-uniform, internal currents no longer cancel.



Using the expression
 $I = Mt$ from above.

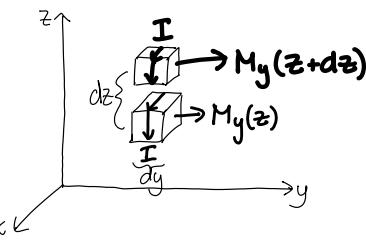
I_x on the boundary btwn volume elements is:

$$I_x = [M_z(y+dy) - M_z(y)] dz = \frac{\partial M_z}{\partial y} dy dz$$

The volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}$$

But we also have to take into account variation in z direction:



This contributes
 $(J_b)_x = - \frac{\partial M_y}{\partial z}$

Putting this together: $(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$