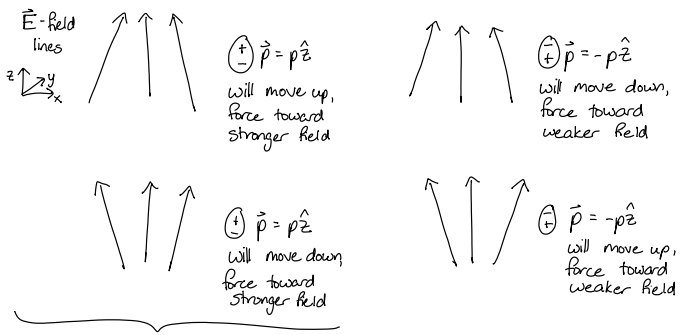
Physics 153 (Hoffman)
Lecture # 11
Thurs, March 5, 2009

Diamagnetism, Paramagnetism

Last time: reviewed $\vec{E}$-fields in matter

materials <u>always</u> polarize along the direction of $\vec{E}$
i.e. $\vec{P}$ is ALWAYS parallel to $\vec{E}$, even if its magnitude is non-linear



This case will prevail for ALL real dielectrics, because $\vec{P}$ will always point in same direction as $\vec{E}$

Magnetic fields in matter:

We found empirically that some materials (e.g. bismuth) move toward weaker field → these are called **diamagnetic** and some materials (e.g. $MnSO_4$) move towards stronger field → these are called **paramagnetic**

Goals for today:

* qualitative understanding of diamagnetism, paramagnetism
* quantitative way to calculate fields in magnetic materials
* the saga of the STM vs. the TEM, continued...

Paramagnets: analogous to polar materials
have unpaired electron spins that act like magnetic dipoles
applied magnetic field rotates these dipoles
but not complete rotation b/c of thermal energy

$$\Rightarrow \vec{M} = \frac{Nm^2}{k_B T}\vec{B}$$

where m is built-in, microscopic moment
and N is # of m's per volume

Diamagnets: analogous to non-polar materials
magnetic field induces a magnetic dipole
(the effect is very weak, but it's present in ALL materials, unlike paramagnetism which is present only in materials with unpaired spins).

Lens' law: will cover in next chapter
idea: electromotive force <u>opposes</u> changing $\vec{B}$-field

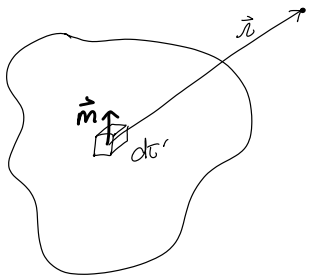
→ induced dipole $\vec{m}$ is opposite applied $\vec{B}$!

⇒ these materials will be repelled by $\vec{B}$-field

Note: diamagnetic effect is always present
but when paramagnetic effect is also present, it dominates.

Field of a magnetized object

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

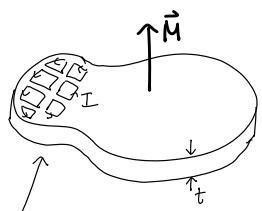


$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

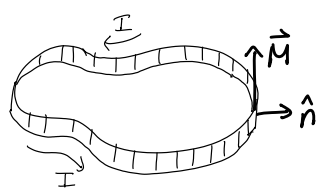
$\downarrow$ math

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \underbrace{\frac{1}{\mathscr{r}}\left[\vec{\nabla}'\times\vec{M}(\vec{r}')\right]d\tau'}_{\substack{\text{volume bound current}\\ \vec{J}_b=\vec{\nabla}\times\vec{M}}} + \frac{\mu_0}{4\pi}\oint \underbrace{\frac{1}{\mathscr{r}}\left[M(\vec{r}')\times d\vec{a}'\right]}_{\substack{\text{surface bound current}\\ \vec{K}_b=\vec{M}\times\hat{n}}}$$

Surface current:



uniform $\vec{M}$
$\Rightarrow$ all internal currents vanish



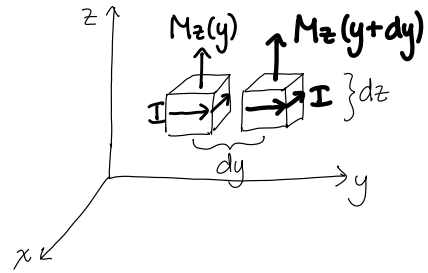
Each tiny loop has thickness t and area a, so in terms of $\vec{M}$ the dipole moment of a single loop is $m = Mat$. But in terms of I, $m = Ia$, so we must have $I = Mt$.
$\Rightarrow K_b = I/t = M$

In terms of the directions of $\vec{M}$ and $\hat{n}$ at the surface, we can write:

$$\vec{K}_b = \vec{M}\times\hat{n}$$

Volume bound current:

When $\vec{M}$ is non-uniform, internal currents no longer cancel.



Using the expression $I = Mt$ from above.

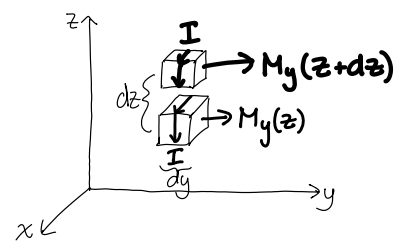
I_x on the boundary btwn volume elements is:

$$I_x = [M_z(y+dy) - M_z(y)]dz = \frac{\partial M_z}{\partial y}dy\,dz$$

The volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}$$

But we also have to take into account variation in z direction:



This contributes

$$(J_b)_x = -\frac{\partial M_y}{\partial z}$$

Putting this together: $(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

Ampere's law

Split the total current $\vec{J}$ into $\vec{J}_f + \vec{J}_b$

the free current, we control

the bound current, resulting from magnetization, we don't control directly

$$\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla}\times\vec{M}$$

$$\vec{\nabla}\underbrace{\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)}_{\text{define } \equiv \vec{H}} = \vec{J}_f$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f} \quad \Rightarrow \quad \boxed{\oint \vec{H}\cdot d\vec{l} = I_{\substack{free,\\ enclosed}}}$$

Example: (Note: this wasn't a well-chosen example, and it wasn't well done in class... here it's at least re-done correctly...)

(a) Infinitely long cylinder carries uniform magnetization $\vec{M}$ parallel to axis $\Rightarrow$ find $\vec{B}$ everywhere.

Solution: no mention of $\vec{J}_f$, so assume $\vec{J}_f = 0$. $\Rightarrow \vec{\nabla}\times\vec{H} = 0$

Also, note that $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = -\vec{\nabla}\cdot\vec{M}$

so $\vec{\nabla}\cdot\vec{H} = 0$ inside, where $\vec{M}$ is constant, and outside, where $\vec{M}=0$.

Inside & outside: $\vec{\nabla}\times\vec{H} = 0$ and $\vec{\nabla}\cdot\vec{H} = 0 \Rightarrow \vec{H} = 0$

What about on the boundary?

$$\vec{\nabla}\cdot\vec{M} = \frac{1}{r}\frac{\partial}{\partial r}(rM_r) + \frac{1}{r}\frac{\partial}{\partial\phi}M_\phi + \frac{\partial}{\partial z}M_z = 0 \Rightarrow \vec{H} = 0 \text{ on boundary too}$$

Inside: $\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0\vec{M}$

Outside: $\vec{B} = \mu_0(\vec{H}+\vec{M}) = 0$

(b) What about an infinitely long rectangular prism?

Solution: same logic still holds.

Check that $\vec{\nabla}\cdot\vec{M} = \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z} = 0$ on boundary here too.

$\Rightarrow$ Inside: $\vec{B} = \mu_0\vec{M}$; Outside: $\vec{B} = 0$

(c) What about a finite cylinder with uniform $\vec{M}$ along axis?

Solution: Still have $\vec{\nabla}\cdot\vec{H} = 0$ inside, outside, and along the sides

But on the ends, we have $\vec{\nabla}\cdot\vec{M} = \partial M_z/\partial z \rightarrow$ diverges $\Rightarrow \vec{H} \neq 0$

There are no free or bound currents on the ends ($\vec{K}_b = \vec{M}\times\hat{n} = 0$ because $\vec{M} \parallel \hat{n}$ on the ends), so we have $\vec{B}_\perp$ continuous across the end boundaries. Therefore $\vec{H}_{\perp(out)} - \vec{H}_{\perp(in)} = \vec{M}_{\perp(in)} - \vec{M}_{\perp(out)} = \vec{M}$.

$\vec{H}$ is continuous across the side boundaries.

Linear Media

For most substances (paramagnetic & diamagnetic), magnetization develops in linear proportion to the field.

So we'd have some headache again:

$\vec{B} \Rightarrow \vec{M} \Rightarrow \vec{B} \Rightarrow \vec{M} \ldots$

But the $\vec{H}$ notation allows us to short-circuit this endless loop to find the self-consistent final $\vec{B}$.

In direct analogy with $\vec{E}$-fields which cause linear polarization $\vec{P}$, we should write: $\vec{M} = \frac{1}{\mu_0}\chi_m\vec{B}$ **X WRONG!**

But the convention is to define

$\boxed{\vec{M} = \chi_m\vec{H}}$ where **χ_m = magnetic susceptibility**

$\Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H}$

$\Rightarrow \boxed{\vec{B} = \mu\vec{H}}$ where **$\mu \equiv \mu_0(1+\chi_m)$ = permeability**

Compare χ_e to χ_m:

	$\varepsilon/\varepsilon_0 = 1+\chi_e$		$\chi_m = \frac{\mu}{\mu_0} - 1$
vacuum	1	hydrogen	-2.2×10^{-9}
helium	1.000065	water	-9×10^{-6}
hydrogen	1.00025	copper	-9.7×10^{-6}
air (dry)	1.00054	gold	-3.4×10^{-5}
water vapor (100°C)	1.00587	bismuth	-1.6×10^{-4}
benzene	2.28	oxygen (gas)	1.9×10^{-6}
diamond	5.7	sodium	8.5×10^{-6}
salt	5.9	aluminum	2.1×10^{-5}
silicon	11.8	tungsten	7.8×10^{-5}
methanol	33	platinum	2.8×10^{-4}
water	80.1	liquid oxygen (-200°C)	3.9×10^{-3}
ice (-30°C)	99	gadolinium	4.8×10^{-1}
$KTaNbO_3$ (0°C)	34,000	"Mu-metal"	~100,000

alloy = 75% Ni, 15% Fe, some Cu, Mo

Conclude: magnetism is typically much weaker that electrical polarization.

Electric fields in matter

Q = charge = "monopole moment"

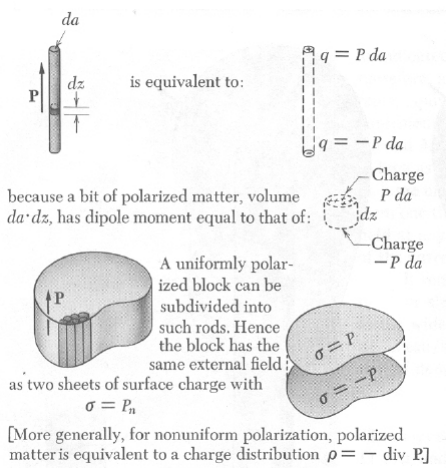
→ potential φ falls off like $1/r$
→ field $\vec{E}$ falls off like $1/r^2$

$\vec{p}$ = "dipole moment" = relative displacement of positive & negative charges

$$= \int_{\substack{\text{volume}\\ \text{of charge}\\ \text{distribution}}} \rho(\vec{r}\,')\vec{r}\,' dv = \sum_{\substack{\text{all point}\\ \text{charges } q_j}} q_j \vec{r}_j$$

→ potential φ falls off like $1/r^2$
→ field $\vec{E}$ falls off like $1/r^3$

$\vec{P}$ = "polarization" = dipole moment per unit volume
(e.g. $N\vec{p}$ where N = # of molecules/cm^3
and $\vec{p}$ = dipole moment of single molecule)



$\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$

$\langle\vec{E}\rangle$ = average $\vec{E}$ inside material = exactly what you would calculate from this simple macroscopic charge distribution picture, ignoring the messy microscopics

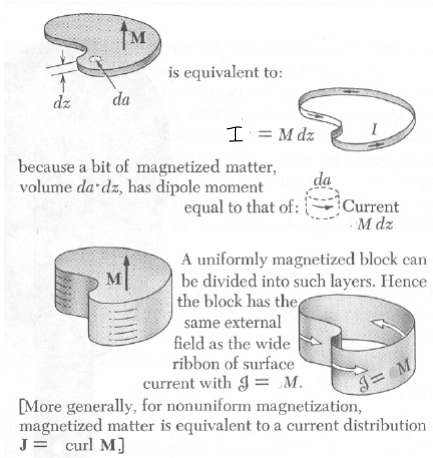
Magnetic fields in matter

(?) = "magnetic monopole" - DOES NOT EXIST, as far as we know

$\vec{m}$ = "dipole moment" = size & strength of current loop
$= Ia$

→ vector potential $\vec{A}$ falls off like $1/r^2$
→ field $\vec{B}$ falls off like $1/r^3$

$\vec{M}$ = "magnetization" = dipole moment per unit volume
(e.g. $N\vec{m}$ where N = # of molecules/cm^3
and $\vec{m}$ = dipole moment of single molecule)



$\vec{J}_{bound} = \vec{\nabla}\times\vec{M}$

$\langle\vec{B}\rangle$ = average $\vec{B}$ inside material = exactly what you would calculate from this simple macroscopic current loop picture, ignoring the messy microscopics

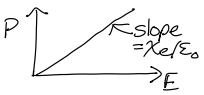
α = "polarizability" = tendency of a non-polar molecule to acquire a dipole moment in an applied $\vec{E}$-field, defined by $\vec{p} = \alpha \vec{E}$

typically $\alpha \sim a^3$ where a = dimension of molecule

χ_e = "dielectric susceptibility" = tendency of a macroscopic material to acquire a macroscopic polarization $\vec{P}$ in the presence of an applied $\vec{E}$ defined by $\vec{P} = \frac{1}{\varepsilon_0}\chi_e \vec{E}$

Note: not all materials obey this nice linear relation, but most do, for reasonable ranges of $\vec{E}$.

For a linear material: $\chi_e = \frac{\varepsilon}{\varepsilon_0} - 1$



2 kinds of materials:

<u>non-polar</u>: molecules have no spontaneous dipole moment, but applied $\vec{E}$-field moves negative charge cloud

- molecule always acquires $\vec{p}$ in the same direction as $\vec{E}$
- relatively weak effect
- exhibited by ALL molecules (molecules that are polar to begin with just have their dipole moment shifted slightly)

<u>polar</u>: molecules each have a dipole moment even when $\vec{E} = 0$, but they are all randomly oriented (cancelling) due to thermal agitation

- $\vec{E}$-field applies torque to each dipole
- dipoles tend to orient with $\vec{p}$ in the same direction as $\vec{E}$
- thermal agitation prevents all from completely aligning

$$\chi_e = \frac{Np^2}{k_B T}$$

- relatively strong effect

Note: both types of materials acquire $\vec{P}$ parallel to $\vec{E}$

"magnetizability" - no letter exists to represent it

χ_m = "magnetic susceptibility" = tendency of a macroscopic material to acquire a macroscopic magnetization $\vec{M}$ in the presence of an applied field $\vec{H}$ defined by $\vec{M} = \chi_m \vec{H}$

2 kinds of materials:

<u>diamagnetic</u>: all electron spins are paired ↑↓
no free spins
entire magnetic moment comes from "orbital" e^- motion
relatively weaker effect

orbital e^- motion changes in applied $\vec{B}$ due to Lenz' law

$$\Delta\vec{m} = \frac{-e^2 r^2}{4m_e}\Delta\vec{B}$$

note the negative sign!
$\Delta\vec{m}$ is opposite the direction of $\Delta\vec{B}$

<u>paramagnetic</u>: some unpaired, "free" electron spins
each spin carries its own spontaneous dipole moment $\hbar e/2m_e$, even when $\vec{B} = 0$
but they are all randomly oriented (cancelling) due to thermal agitation

- $\vec{B}$-field applies torque to each dipole
- dipoles tend to orient with $\vec{m}$ in the same direction as $\vec{B}$
- thermal agitation prevents all from completely aligning

$$\chi_{pm} = \frac{Nm^2}{k_B T}$$

- relatively strong effect

Note: all materials acquire a small $\vec{M}$ anti-parallel to $\vec{B}$, from the diamagnetic effect
some materials acquire an additional, stronger $\vec{M}$ parallel to $\vec{B}$ from the paramagnetic effect, which totally drowns out the weak diamagnetic effect

ε = "dielectric constant" - property of a material such that:

① if material is stuck between capacitor plates, the charge on the plates is increased from vacuum value Q_0 to $Q = \varepsilon Q_0$

② more generally, electric fields produced by free charges in the material are reduced by a factor ε with respect to vacuum

For a linear material, $\varepsilon = \varepsilon_0(\chi_e + 1)$

$\vec{D}$ = "displacement current" $\equiv \varepsilon_0 \vec{E} + \vec{P}$

For a linear material, this reduces to

$$\varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1+\chi_e)\vec{E} = \varepsilon \vec{E}$$

$\vec{D}$ is a convenient definition that allows us to simplify Maxwell's laws in matter.

You can <u>sort of</u> think of $\vec{D}$ as the "field produced by the free charges" but it's not a perfect analogy because $\vec{\nabla} \times \vec{D} \neq 0$

Note: $\vec{D}$ is not a very useful concept, because usually in the lab we do actually control $\vec{E}$ directly, because we can easily apply voltages with batteries or power supplies. If we control V, then $\vec{E}$ follows directly from $\vec{\nabla} V$, regardless of what material is there.

It's not actually so easy to control the free charge in the laboratory (only in our theoretical classroom minds).

Gauss's law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \quad \longrightarrow \quad \vec{\nabla} \cdot \vec{D} = \rho_{free}$

Energy/work: $U = \frac{\varepsilon_0}{2} \int E^2 d\tau \quad \rightarrow \quad W = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, d\tau$

μ = "magnetic permeability" = property of a material such that the fundamental magnetic field $\vec{B}$ differs from the applied magnetic field $\vec{H}$ by $\vec{B} = \mu \vec{H}$. Also, $\mu = \mu_0(1+\chi_m)$

(This only makes sense for linear materials.)

$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}$

For a linear material, this reduces to

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \chi_m \vec{H} \Rightarrow \vec{B} = \mu_0 (1+\chi_m) \vec{H} = \mu \vec{H}$$

$\vec{H}$ is a convenient definition that allows us to simplify Maxwell's laws in matter

You can <u>sort of</u> think of $\vec{H}$ as the "field produced by the free currents" but it's not a perfect analogy because $\vec{\nabla} \cdot \vec{H} \neq 0$.

Note: $\vec{H}$ is a much more useful concept than $\vec{D}$, because usually in the lab we do actually control $\vec{H}$ directly, because we do supply and control the free currents.

It's this practical difference (the ease of controlling $\vec{E}$ directly but only controlling $\vec{B}$ indirectly via $\vec{H}$) that leads to the annoying asymmetry in the definitions:

$\vec{P} = \frac{1}{\varepsilon_0} \chi_e \vec{E}$	$\vec{M} = \chi_m \vec{H}$
$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$	$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$
$\varepsilon = \varepsilon_0 (1+\chi_e)$	$\mu = \mu_0 (1+\chi_m)$
$\vec{D} = \varepsilon \vec{E}$	$\vec{B} = \mu \vec{H}$

Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J}_{free}$

Energy/work: $U = \frac{1}{2\mu_0} \int B^2 d\tau \quad \rightarrow \quad W = \frac{1}{2} \int \vec{B} \cdot \vec{H}\, d\tau$

STM vs TEM saga, continued...

If you want to get rid of an unwanted electric field that's messing up your experiment, how do you do it?

Answer: screen it with a conducting shield
A perfect conductor in DC limit has $\varepsilon \to \infty$
in other words it can "polarize" infinitely well to screen $\vec{E}$.

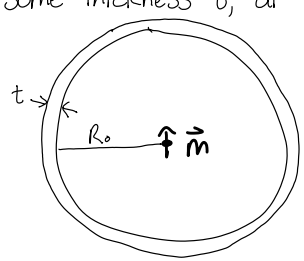
But there's no magnetic analog to a perfect conductor
(unless you consider superconductors at very low T).
In fact, there are very few magnetic materials with
χ_m even approaching values like χ_e of typical dielectrics.
Magnetism in materials is weak in comparison.

Best bet: material called "Mu metal"
(because it has large $\mu/\mu_0 \sim 10^5$)
~75% Ni, 15% Fe, some copper & molybdenum

Problem: hard to work with, expensive
must be heat-treated (annealed in H_2 at high T)
in exact final shape or loses its high μ

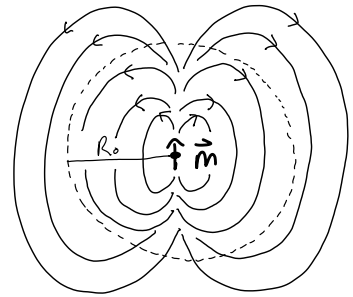
Let's consider my 9T solenoid magnet to be a simple dipole
$\vec{m} = 8470\hat{z}$ A·m², located at the origin
(as we computed, last class)

Now consider surrounding this with a spherical shell of
μ-metal of some thickness t, at radius $R_0 = 10$ m.

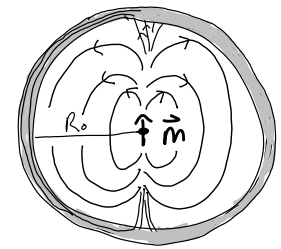


① Qualitatively, what happens?

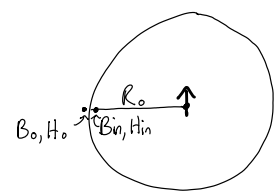
without μ-metal present:



with μ-metal present:
return field lines concentrated in μ-metal



② Order of magnitude, how thick should μ-metal be
to reduce the equatorial field at 10 m from 8.5 mG
(we calculated last time) to 0.1 mG (TEM specs)?



Consider the fields on the "equator" for r just < 10m
(just at the edge of the air of the room, where $\mu = \mu_0$)
and r just > 10m (just within the μ-metal shield).

Assume to 1st order that $\vec{B}$ isn't much perterbed within the room itself.

$$\Rightarrow \vec{B}_{in} = 8.5 \text{ mG} = \mu_0 \vec{H}_{in}$$

But within the μ-metal, we have $\vec{B}_0 = \mu \vec{H}_0$
And we have a boundary condition $\vec{H}_{in} = \vec{H}_0$ because there's no free current at the boundary, so an Amperian loop straddling the boundary must have $\oint \vec{H} \cdot d\vec{l} = 0$

$$\Rightarrow \vec{B}_o = \mu \vec{H}_o = \mu \vec{H}_{in} = \mu\left(\frac{1}{\mu_o}\vec{B}_{in}\right) = \frac{\mu}{\mu_o}\vec{B}_{in} \approx 10^5\,(8.5\,mG)$$

Flux lines can never end (no magnetic monopoles)

Without the μ-metal, the total flux that needs to return outside R_o is:

$$\Phi_1 = \int B\,da = \int_{R_o}^{\infty} dr \int_0^{2\pi} r\,d\phi\, B = 2\pi \int_{R_o}^{\infty} r\,dr\, \frac{B_{in} R_o^3}{r^3} = 2\pi B_{in} R_o^2$$

With μ-metal, we need the same amount of total flux return but now we have the much stronger $B_o \approx 10^5 B_{in}$ so we need much less area for it to return through.

$$\Phi_2 \approx 2\pi \int_{R_o}^{R_1} r\,dr\, \frac{B_o R_o^3}{r^3} = 2\pi B_o R_o^2\left(1 - \frac{R_o}{R_1}\right)$$

Set $\Phi_1 = \Phi_2$:

$$2\pi B_{in} R_o^2 = 2\pi B_o R_o^2\left(1 - \frac{R_o}{R_1}\right)$$

$$B_{in} = B_o\left(1 - \frac{R_o}{R_1}\right)$$

$$\frac{R_o}{R_1} \approx 1 - 10^{-5}$$

$$R_o(1 + 10^{-5}) \approx R_1$$

$$\Rightarrow t = R_1 - R_o = 10^{-5} R_o = 10^{-5}(10m) = 0.1\,mm$$

③ How much does this cost ($\$20/ft^2$)

$$4\pi(10m)^2 = 13{,}521\ ft^2 = \$270{,}000$$

Doesn't include shipping, installation, shipping back for re-annealing, re-shipping, re-installing...
Also, uses an overly idealized model.
In fact, real μ-metal is non-linear, and $\mu \ll 10^5$ for small fields.