Physics 153 (Hoffman)
Lecture #12
Tues, March 10, 2009

Midterm Review

List of types of problems:

(2.15) Given a high-symmetry charge distribution, find $\vec{E}$
Use Gauss' law

(2.26) Given a charge distribution, find φ
Usually, just integrate: $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}\,d\tau'$

Can then compute $\vec{E} = -\vec{\nabla}\varphi$

→ (2.40) Find the capacitance of 2 conductors (or 1 conductor w.r.t. infinity)
Usually, just put $\pm Q$ on conductors, find the potential φ, compute $\Delta\varphi$ btwn 2 conductors and compute $C = Q/\Delta\varphi$
(Sometimes, especially if dielectrics are present, it's easier to start by assuming a potential difference $\Delta\varphi$ then find Q)

(3.35) Given a charge distribution and a conductor, find $\varphi, \vec{E}$ etc.
Use the method of images to place strategic fictitious charges on the other side of the conductor.

(3.37) Laplace's eqn in spherical coordinates
→ (3.39) " cylindrical
(3.47) " cartesian

(3.30) Compute the multipole expansion of a charge distribution

(3.49, 4.9) Compute the force on a dipole, or force dipole exerts

(4.2) Compute the polarizability of a charge distribution
Balance the force from the applied field against the restoring force

(4.13) Compute $\vec{E}$ from an object with polarization $\vec{P}$
Find $\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = -\vec{\nabla}\cdot\vec{P}$ and find φ directly (like 2.26)

→ (4.18) Given a high-symmetry dielectric + conductor situation, find $\vec{E}$
Use Gauss' law $\vec{\nabla}\cdot\vec{D} = \sigma_{free}$ and $\vec{D} = \varepsilon\vec{E}$

(4.28) Compute the force on a dielectric

(5.3) Force on a moving charge in $\vec{B}$

① → (5.8, 5.13) Find $\vec{B}$ from a current distribution
(a) if no symmetry, need to integrate Biot-Savart
(b) if symmetry, use Ampere's law

(5.22) Find the magnetic vector potential $\vec{A}$ of a current distribution

② → (5.35) Find the magnetic dipole moment $\vec{m}$ of a current distribution

Goals for today: * wrap up STM vs. TEM saga (magnetization, ch6)
* quantum dot

STM vs TEM saga, continued...

If you want to get rid of an unwanted electric field that's messing up your experiment, how do you do it?

Answer: screen it with a conducting shield
A perfect conductor in DC limit has $\varepsilon \to \infty$
in other words it can "polarize" infinitely well to screen $\vec{E}$.

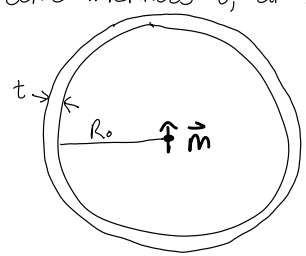
But there's no magnetic analog to a perfect conductor
(unless you consider superconductors at very low T).
In fact, there are very few magnetic materials with χ_m even approaching values like χ_e of typical dielectrics.
Magnetism in materials is weak in comparison.

Best bet: material called "Mu metal"
(because it has large $\mu/\mu_0 \sim 10^5$)
~75% Ni, 15% Fe, some copper & molybdenum

Problem: hard to work with, expensive
must be heat-treated (annealed in H_2 at high T)
in exact final shape or loses its high μ

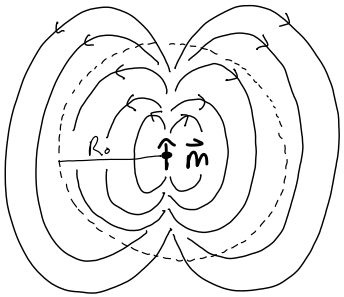
Let's consider my 9T solenoid magnet to be a simple dipole
$\vec{m} = 8470\hat{z}$ A·m², located at the origin
(as we computed, last class)

Now consider surrounding this with a spherical shell of μ-metal of some thickness t, at radius $R_0 = 10$ m.

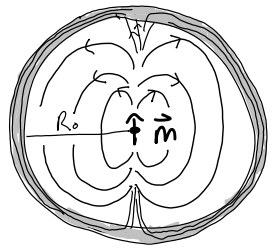


(a) Qualitatively, what happens?

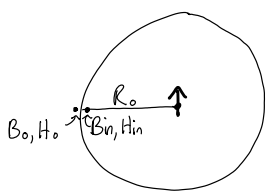
without μ-metal present:



with μ-metal present:
return field lines concentrated in μ-metal



(b) Order of magnitude, how thick should μ-metal be to reduce the equatorial field at 10 m from 8.5 mG (we calculated last time) to 0.1 mG (TEM specs)?



Consider the fields on the "equator" for r just $< 10m$ (just at the edge of the air of the room, where $\mu = \mu_0$) and r just $> 10m$ (just within the μ-metal shield).

Assume to 1st order that $\vec{B}$ isn't much perterbed within the room itself.

$$\Rightarrow \vec{B}_{in} = 8.5\ mG = \mu_0 \vec{H}_{in}$$

But within the μ-metal, we have $\vec{B}_0 = \mu \vec{H}_0$
And we have a boundary condition $\vec{H}_{in} = \vec{H}_0$ because there's no free current at the boundary, so an Amperian loop straddling the boundary must have $\oint \vec{H} \cdot d\vec{l} = 0$

$$\Rightarrow \vec{B}_0 = \mu \vec{H}_0 = \mu \vec{H}_{in} = \mu \left(\frac{1}{\mu_0} \vec{B}_{in} \right) = \frac{\mu}{\mu_0} \vec{B}_{in} \approx 10^5 (8.5\ mG)$$

Flux lines can never end (no magnetic monopoles)

Without the μ-metal, the total flux that needs to return outside R_0 is:

$$\Phi_1 = \int B\, da = \int_{R_0}^{\infty} dr \int_0^{2\pi} r\, d\phi\, B = 2\pi \int_{R_0}^{\infty} r\, dr\, B_{in} \frac{R_0^3}{r^3} = 2\pi B_{in} R_0^2$$

With μ-metal, we need the same amount of total flux return but now we have the much stronger $B_0 \approx 10^5 B_{in}$ so we need much less area for it to return through.

$$\Phi_2 \approx 2\pi \int_{R_0}^{R_1} r\, dr\, B_0 \frac{R_0^3}{r^3} = 2\pi B_0 R_0^2 \left(1 - \frac{R_0}{R_1}\right)$$

Set $\Phi_1 = \Phi_2$:

$$2\pi B_{in} R_0^2 = 2\pi B_0 R_0^2 \left(1 - \frac{R_0}{R_1}\right)$$

$$B_{in} = B_0 \left(1 - \frac{R_0}{R_1}\right)$$

$$\frac{R_0}{R_1} \approx 1 - 10^{-5}$$

$$R_0 (1 + 10^{-5}) \approx R_1$$

$$\Rightarrow t = R_1 - R_0 = 10^{-5} R_0 = 10^{-5} (10m) = 0.1\ mm$$

(c) How much does this cost ($\$20/ft^2$)

$$4\pi (10m)^2 = 13{,}521\ ft^2 = \$270{,}000$$

Doesn't include shipping, installation, shipping back for re-annealing, re-shipping, re-installing...
Also, uses an overly idealized model.
In fact, real μ-metal is non-linear, and $\mu \ll 10^5$ for small fields.

How else can we solve this problem?

Midterm review ①, ②: set up a second solenoid such that the dipole moments cancel, so far field falls off like $1/r^4$ but the central fields still sum to 9T

Reminder: $\vec{B}$ on axis of a current loop:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\ell' = \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}\hat{z}$$

$\vec{B}$ at center of solenoid:

$$\vec{B} = \frac{\mu_0 I N}{2}\frac{1}{(H^2+R^2)^{1/2}}\hat{z}$$

$\vec{m}$ of solenoid $= \pi R^2 N I \hat{z}$

So now we need $R_1 = 2.5$ cm, $R_2 > R_1$ but as small as possible
$H_1 = 30$ cm, take $H_2 = H_1$ (bigger H means better central homogeneity, but we don't have space to make $H_2 > H_1$)
N_1, N_2 as small as possible (wire is expensive!)
$I_1 = I_2 = 80$ A (can't exceed critical current of superconductor)

② $\pi R_1^2 N_1 I_1 = \pi R_2^2 N_2 I_2$
$\Rightarrow R_1^2 N_1 = R_2^2 N_2$

① $$\frac{\mu_0 I_1 N_1}{2}\frac{1}{(H_1^2+R_1^2)^{1/2}} - \frac{\mu_0 I_2 N_2}{2}\frac{1}{(H_2^2+R_2^2)^{1/2}} = B_{tot} = 9 \text{ Tesla}$$

$$N_1 - N_2\frac{(0.301\text{ m})}{[(0.3\text{m})^2+R_2^2]^{1/2}} = 179\,049$$

Let's set $R_2 = 5$ cm $\Rightarrow N_1 = 179\,049 + N_2\left(\frac{0.301\text{ m}}{0.304\text{ m}}\right)$
$= 179\,049 + 0.99 N_2$

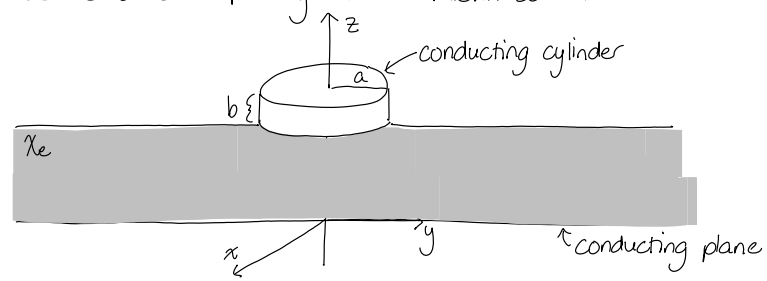
From dipole equality, $N_2 = \frac{1}{4}N_1$
$\Rightarrow 0.7525 N_1 = 179\,049$

$N_1 = 237,938$ $\quad N_2 = 59,485$

~5x more turns than before

Quantum Dot (midterm review ③, ④, and ⑤)

What is the capacitance of a cylindrical quantum dot of radius a and thickness b, separated from an infinite conducting plane by a dielectric of susceptibility χ_e and thickness h?



Solution: Apply a charge Q to the cylinder,
solve Laplace's equation in cylindrical coords to find φ everywhere,
compute $\Delta\varphi$ between cylinder and plane,
and compute $C = Q/\Delta\varphi$.

In order to solve $\nabla^2\varphi = 0$, we need boundary conditions.
The applied Q on the dot will induce some charge on the dielectric and conducting planes, but we know this charge will be on order $-Q$, and will be finite in extent.

① φ is continuous across all boundaries
② $\varphi \to 0$ as $s \to \infty$ and $z \to \infty$
③ $\varphi = \varphi_0 =$ constant on the dot
④ $\varphi =$ constant on the conducting plane

Comparing ② and ④, we can just set $\varphi = 0$ on the conducting plane (and, in fact, for all $z<0$; we also know $\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}\varphi = 0$ for $z<0$)

Let $\varphi_1(s,z)$ be the potential in the vacuum above the dielectric.
Let $\varphi_2(s,z)$ be the potential in the dielectric.

Let $\sigma_1(s,z)$ be the free charge density on the dot.

$$\int_0^a \sigma_1(s,h)\,2\pi s\,ds + \int_0^a \sigma_1(s,h+b)\,2\pi s\,ds + \int_h^{h+b}\sigma_1(a,z)\,2\pi a\,dz = Q$$

Let $\sigma_2(s)$ be the free charge density on the conducting plane.
If $\varphi \to 0$ at ∞, then $\vec{E} \to 0$ at ∞, and $\vec{D} = \varepsilon\vec{E}$ must $\to 0$ at ∞.

Across a boundary, the discontinuity in $D^\perp$ is σ_{free}.

5 top of the dot:

$$D_1^\perp = -\varepsilon_0 \frac{\partial \varphi_1}{\partial z}\Big|_{z=b+h} = \sigma_1(s, b+h)$$

6 side walls of the dot:

$$D_1^\perp = -\varepsilon_0 \frac{\partial \varphi_1}{\partial s}\Big|_{s=a} = \sigma_1(a, z)$$

7 top of the dielectric

$$s<a: \quad -D_2^\perp = \varepsilon \frac{\partial \varphi_2}{\partial z}\Big|_{z=h} = \sigma_1(s,h)$$

$$s>a: \quad D_1^\perp - D_2^\perp = -\varepsilon_0 \frac{\partial \varphi_1}{\partial z}\Big|_{z=h} + \varepsilon \frac{\partial \varphi_2}{\partial z}\Big|_{z=h} = 0$$

8 conducting plane

$$D_2^\perp = \varepsilon \frac{\partial \varphi_2}{\partial z}\Big|_{z=0} = \sigma_2(s)$$

Laplace's eqn in cylindrical coordinates:

$$\nabla^2 \varphi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

Let $\varphi(s,z) = S(s)Z(z)$

$$\frac{\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)}{S(s)} + \underbrace{\frac{\frac{\partial^2 Z}{\partial z^2}}{Z(z)}}_{\frac{\partial^2 Z}{\partial z^2} = (\text{const})Z} = 0$$

We know $Z(z) \to 0$ as $z \to \infty$, so we want this to have an exponential solution (not sin or cos), so choose const >0.

$$\Rightarrow Z(z) = Ae^{kz} + Be^{-kz}$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) = -kS$$

Check formula sheet: $S(s) = C_k s^k + D_k s^{-k} \quad (k>0)$
or $C_0 \ln s + D_0 \quad (k=0)$

$\varphi_1(s,z) \to 0$ as $z \to \infty$, so we need $A=0$

$\varphi_1(s,z) \to 0$ as $s \to \infty$, so we need $C_0 = D_0 = C_k = 0$

$$\Rightarrow \varphi_1(s,z) = \sum_{k=1}^{\infty} \frac{D_k}{s^k} e^{-kz}$$

$\varphi_2(s,z) \to 0$ as $s \to \infty$, so we need $C_0 = D_0 = C_k = 0$

$$\Rightarrow \varphi_2(s,z) = \sum_{k=1}^{\infty} \frac{1}{s^k}\left(A_k e^{kz} + B_k e^{-kz}\right)$$

Boundary condition 4 gives us, for all s:

$$\varphi_2(s,0) = \sum_{k=1}^{\infty} \frac{1}{s^k}(A_k + B_k) = 0 \quad \Rightarrow \quad B_k = -A_k$$

$$\Rightarrow \varphi_2(s,z) = \sum_{k=1}^{\infty} \frac{A_k}{s^k}\left(e^{kz} - e^{-kz}\right)$$

Boundary condition 7 gives us

$$s>a: \quad -\varepsilon_0 \sum_{k=1}^{\infty} \frac{D_k}{s^k}(-k)e^{-kh} + \varepsilon \sum_{k=1}^{\infty} \frac{A_k}{s^k}\left(ke^{kh} + ke^{-kh}\right)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{ke^{-kh}}{s^k}\left[D_k + \frac{\varepsilon}{\varepsilon_0}A_k(1+e^{2kh})\right] = 0$$

$$s<a: \quad \varepsilon \sum_{k=1}^{\infty} \frac{A_k}{s^k}\left(ke^{kh} + ke^{-kh}\right) = \sigma_1(s)$$

Boundary condition 1 gives us

$$s>a: \quad \sum_{k=1}^{\infty} \frac{D_k}{s^k}e^{-kh} = \sum_{k=1}^{\infty} \frac{A_k}{s^k}\left(e^{kh} - e^{-kh}\right)$$

$$s<a: \quad \sum_{k=1}^{\infty} \frac{A_k}{s^k}\left(e^{kh} - e^{-kh}\right) = \varphi_0$$

Answer: $C = 4\varepsilon_0(\varepsilon+1)a$ (in the limit $a \gg b$)