Physics 153 (Hoffman)
Lecture #13
Tues, March 17, 2009

Happy St. Patrick's Day

Electrodynamics

Where are we? We've covered all of electrostatics and magnetostatics, both in vacuum and in materials. We know how stationary charges produce $\vec{E}$-fields and steady state currents produce $\vec{B}$-fields. We know the forces exerted by these fields on charges.

In summary, we have:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \qquad \vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

Now it's time to look at dynamics: how do we determine $\vec{E}$ and $\vec{B}$ from moving source charges and non-steady-state currents?

Goals for today:

* Ohm's law
* EMF (electromotive force)
* Faraday's & Lenz' laws
* Inductance → energy stored in B-fields
* Maxwell's equations in complete form

Ohm's Law

Most familiar: $V = IR$ (global equation of a circuit element)

More general: $\vec{J} = \sigma\vec{E}$ (local equation)

where σ = conductivity is an intrinsic property of a substance
$\rho = 1/\sigma$ = resistivity is also an intrinsic property
but R = resistance is an extrinsic property which depends both on the type of material and on its geometry

ρ ranges from $10^{-8}\ \Omega\text{-m}$ for copper to 10^{16} for quartz

Question: We said that $\vec{E} = 0$ inside a conductor, so how do we ever get any $\vec{J}$ flowing?

First guess: current flows only along surface?
No, because we said $\vec{E}^{\parallel} = 0$ (parallel to surface) as well.

Answer: For any conductor with finite conductance (any material at room temperature) there is a very small $\vec{E}$ field within, when $\vec{J}$ is flowing. For example, a 1mm diameter copper wire carrying 10 mA of current would have:

$$E = \frac{1}{\sigma}J = (10^{-8}\ \Omega\text{-m})\left(\frac{0.01\ \text{A}}{\pi(0.001/2)^2}\right) = 1.3\times10^{-4}\ \text{V/m}$$

Compare to the field across a typical 100 pF capacitor in a 1cm package with $\kappa = \varepsilon/\varepsilon_0 = 100$ and 10 V across it:

$$C = \frac{\varepsilon A}{d}\ ;\ E = \frac{\varphi}{d} = \frac{\varphi C}{\varepsilon A} = \frac{(10\text{V})(10^{-10}\ \text{F})}{(100\times8.85\times10^{-12}\ \text{F/m})(0.01\text{m})^2} = 1.13\times10^{4}\ \text{V/m}$$

And compare to the breakdown field of air: 3×10^{6} V/m
So mostly we just neglect $\vec{E}$ in conductors.
In fact, in superconductors (e.g. Nb below 9K) we really do have $\vec{E} = 0$.

Question: What happens if we try to apply a large $\vec{E}$ to conductor?

Answer: if conductor is isolated, charge redistributes to cancel $\vec{E}$ (thus causing a very large $\vec{J}$ for a very short duration).
If conductor is not isolated, very large $\vec{J}$ will flow but $P = IV$ will be very large → possibly burn up conductor.

Question: What about large $\vec{E}$ to superconductor?

Answer: Can't. Inevitably, the voltage source will have more series resistance than the superconductor, so the field will develop inside voltage source rather than inside superconductor.

Question: If the force on a charged particle is $\vec{F} = m\vec{a} = Q\vec{E}$ i.e. $\vec{a} \propto \vec{E}$ then how in the world can we claim $\vec{J} = nQ\vec{v} = \sigma\vec{E}$, i.e. $\vec{v} \propto \vec{E}$???

(n: density of particles)

Answer: the $\vec{v}$ appearing in $\vec{J} = nQ\vec{v}$ is the "drift velocity". In fact the actual thermal velocity of any one electron is much larger than this, but it is in a random direction. In the time interval between collisions, the electron starts to pick up a component of velocity in accordance with $\vec{F} = Q\vec{E}$ but every time the electron suffers a collision, its velocity direction is re-randomized.

ℓ = mean free path

$\tau = \ell / v_{thermal}$ = time between collisions

$$v_{avg} = \frac{1}{2} a\tau = \frac{1}{2}\frac{QE}{m}\frac{\ell}{v_{thermal}}$$

$$J = nQv_{avg} = \frac{nQ^2\ell}{2mv_{thermal}} E$$

$$\Rightarrow \sigma = \frac{nQ^2\ell}{2mv_{thermal}}$$ "Drude model" (classical model for conductivity)

Question: How do we get from $J = \sigma E$ to $V = IR$?

Answer: for a piece of material of length d and cross-section A:

$V = Ed$

$I = JA$

So $J = \sigma E \Rightarrow I/A = \sigma V/d \Rightarrow V = \underbrace{d/(A\sigma)}_{\text{call this geometry-dependent constant } R} I$

Power = (work done/charge)(charge/time) = $VI = I^2R$

Question: how do charges far from the source know to move? (And apparently they know it near-instantaneously, because the lightbulb turns on as soon as you flip the switch.)

Answer: there must be an additional small $\vec{E}$ which develops throughout the loop.

$\vec{f} = \vec{f}_s + \vec{E}$ ← extra $\vec{E}$ that develops

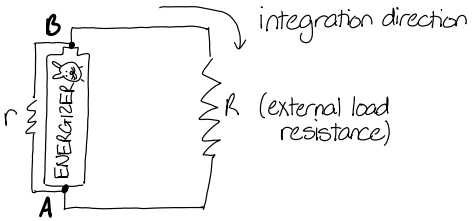
($\vec{f}$: total force per unit charge; $\vec{f}_s$: force per unit charge from the source (usually confined to one part of loop, e.g. battery))

Electromotive Force (EMF): $\mathcal{E} = \oint \vec{f} \cdot d\vec{\ell}$

Convince yourself that if $\mathcal{E} \neq 0$ then there is a net motion of charge around the loop, but if $\mathcal{E} = 0$ then there is no net motion.

Battery:

(reality: the source always has some finite resistance)

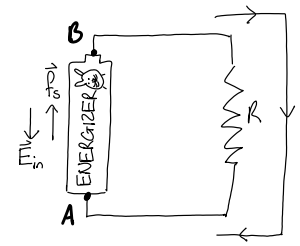


$$\mathcal{E} = \oint \vec{f} \cdot d\vec{\ell} = \oint (\vec{f}_s + \vec{E}) \cdot d\vec{\ell} = \underbrace{\oint \vec{f}_s \cdot d\vec{\ell}}_{= \int_A^B \vec{f}_s \cdot d\vec{\ell}} + \underbrace{\oint \vec{E} \cdot d\vec{\ell}}_{=0 \text{ for static } \vec{E}}$$

In the ideal case in which $r \to 0$, the net force on the charges within the battery must be zero, so $\vec{f} = \vec{f}_s + \vec{E} = 0 \Rightarrow \vec{f}_s = -\vec{E}_{in}$ so the voltage across the battery is

$$\Delta V = V_B - V_A = -\int_\infty^B \vec{E}_{in} \cdot d\vec{\ell} - \left(-\int_\infty^A \vec{E}_{in} \cdot d\vec{\ell}\right) = -\int_A^B \vec{E}_{in} \cdot d\vec{\ell} = \int_A^B \vec{f}_s \cdot d\vec{\ell} = \mathcal{E}$$

$|\vec{E}_{in}| = |\vec{f}_s|$



some small $\vec{E}_{out}$ (not necessarily constant around the loop) such that

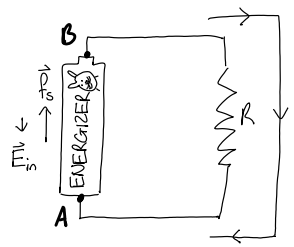
$$\oint \vec{E} \cdot d\vec{\ell} = \underbrace{\int_A^B \vec{E}_{in} \cdot d\vec{\ell}}_{\text{in battery}} + \underbrace{\int_B^A \vec{E}_{out} \cdot d\vec{\ell}}_{\text{around loop}} = 0$$

In the realistic case in which $r \neq 0$ (but $r \ll R$) there must be some net upward force on the charges within the resistor, so $\vec{E}_{in}$ still points down but now $|\vec{E}_{in}| < |\vec{f}_s|$.
(Question: why can't $\vec{E}_{in}$ point up?
Answer: because then we could never have $\oint \vec{E} \cdot d\vec{l} = 0$)

$$\Delta V = V_B - V_A = -\int_\infty^B \vec{E}_{in} \cdot d\vec{l} - \left(-\int_\infty^A \vec{E}_{in} \cdot d\vec{l}\right) = -\int_A^B \vec{E}_{in} \cdot d\vec{l} < \int_A^B \vec{f}_s \cdot d\vec{l} = \mathcal{E}$$

So the presence of the internal resistance r causes the actual voltage difference across the battery terminals to sag slightly below the EMF of the battery: $\Delta V < \mathcal{E}$

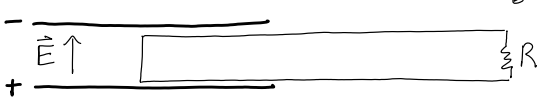
$|\vec{E}_{in}| < |\vec{f}_s|$



some small $\vec{E}_{out}$ (not necessarily constant around the loop) such that

$$\oint \vec{E} \cdot d\vec{l} = \underbrace{\int_A^B \vec{E}_{in} \cdot d\vec{l}}_{\text{in battery}} + \underbrace{\int_B^A \vec{E}_{out} \cdot d\vec{l}}_{\text{around loop}} = 0$$

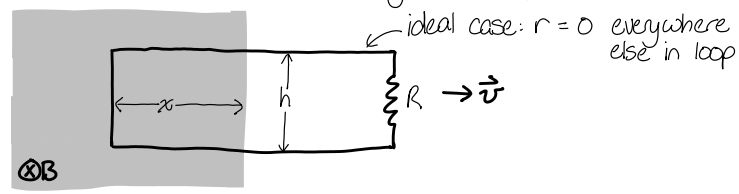
Capacitor: What's the EMF around this loop?



We must have $\oint \vec{E} \cdot d\vec{l} = 0$
We can't have net current flowing around the loop or we would have a perpetual motion machine.
Charge must redistribute around the left half of the loop to cancel the apparent $\vec{E}$ field within the wire itself so that $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 0$

Note: EMF is instantaneous
It's just defined that way: it's a snapshot of $\oint \vec{E} \cdot d\vec{l}$ for the simultaneous $\vec{E}$ everywhere on the loop, <u>not</u> the $\vec{E}$ encountered by a single charge as it makes its way around the loop in some finite amount of time.

Generator: pull a loop to the right in the presence of $\vec{B}$



$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = vBh$$

magnetic flux $\equiv \Phi = \int \vec{B} \cdot da$

For this particular loop, $\Phi = Bhx$

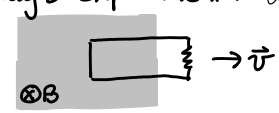
$$\Rightarrow \frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

Putting this together gives $\mathcal{E} = -\dfrac{d\Phi}{dt}$

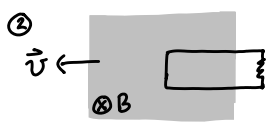
In fact, this is generally true for any shape loop and any (even inhomogenous) $\vec{B}$-field.
(General proof is a little tedious, see pages 296-298 in Griffiths.)

Faraday's experiment: Which way does I flow?

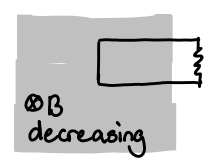
①



②



③



All 3 cases: I flows clockwise

But in case ① we have a magnetic force setting up the EMF whereas in cases ② and ③ we have no moving charges so we can have no magnetic force → it must be an electric force to set up an EMF on stationary charges.

Faraday's Law: changing $\vec{B}$ induces $\vec{E}$

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\Downarrow \text{(Stokes' thm)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

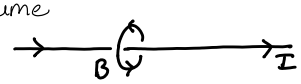
Lenz's Law: nature abhors changing $\vec{B}$

(this is just the same as above, but it's easier to get the signs right)

Example: An infinitely long straight wire (along $\hat{z}$) carries a slowly varying current $I(t)$. Find $\vec{E}$ as a function of distance s from the wire.

Solution: If $I(t)$ is changing slowly, let's assume

$B = \frac{\mu_0 I}{2\pi s}$ circling around the wire:



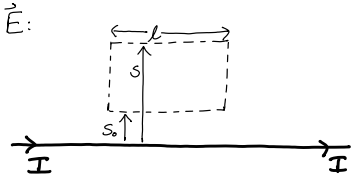
Induced $\vec{E}$ is going to run parallel to I, trying to oppose changes in I, because Lenz says that nature doesn't want $\vec{B}$ to change.

We can employ a sort of "Ampere's law" for $\vec{E}$, where $-d\Phi/dt$ is the source term:

original Ampere's law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

analog to Ampere's law for $\vec{E}$: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_{enclosed}$

Draw an "Amperian loop" for $\vec{E}$:



$$\oint \vec{E}\cdot d\vec{l} = E(s_0)l - E(s)l = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$= -\frac{\mu_0}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{l}{s'}ds = -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\left[\ln s - \ln s_0\right]$$

$$\Rightarrow \vec{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{z}$$

for some K which doesn't depend on s (but may depend on t)

What's wrong here?

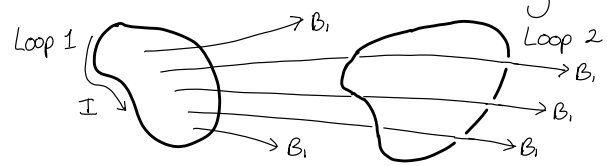
$\vec{E}(s)$ can't really blow up as $s\to\infty$

We can't make this so-called "quasistatic approximation" that $B = \frac{\mu_0 I}{2\pi s}$ because s only knows about the current I at a time $t = s/c$ later.

Moral of the story: quasistatic approximation often works to calculate instantaneous $\vec{B}$ from instantaneous $\vec{J}$ to get the right side of $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

but it fails when we start talking about infinite objects or very long distances.

Mutual Inductance: consider two conducting loops



A current I around Loop 1 will produce a field $\vec{B}_1$, some of which penetrates Loop 2.

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{(Biot-Savart)}$$

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2 \propto I_1$$

$\Rightarrow \Phi_2 = M_{21} I_1$ where M_{21} comes from the messy double integral:

$$M_{21} = \frac{\mu_0}{4\pi}\int\oint \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}\cdot d\vec{a}_2$$

We can write M_{21} in a "cuter" way (not necessarily easier to calculate, but will give us better intuition) as follows.

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = \int (\vec{\nabla}\times\vec{A}_1)\cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$$

$$= \oint \left(\frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1}{\mathscr{r}} \right) \cdot d\vec{l}_2 = \frac{\mu_0}{4\pi} I_1 \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}$$

$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}$$

Which tells us that:

① M_{21} is purely a geometric quantity

② $M_{21} = M_{12}$, so drop the subscripts and call it M

If you now vary the current I_1:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_2}{dt}$$

where here we are again implicitly using the quasistatic approximation, hidden inside the use of the Biot-Savart law to calculate M. Here, the quasistatic approximation is probably valid, as long as the loops are not too far apart and the current does not vary too quicky.

From a practical standpoint, it is challenging to make a current in a loop vary too quickly, because...

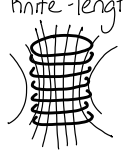
Self Inductance: in fact, a current loop is threaded by its own $\vec{B}$-field in proportion to its own current I

$$\Phi = LI \quad \Rightarrow \quad \mathcal{E} = -L\frac{dI}{dt}$$

L = inductance, measured in Henries = Volt-second/Ampere

Example: Consider again my 9 Tesla magnet, with $I = 80$ Amps, diameter 5 cm, height 30 cm, and $N = 53{,}900$ turns. With a 5V power supply, approximately how fast can we ramp the field from 0 Tesla to 9 Tesla?

Solution: Exact self-inductance is going to be pretty tricky to calculate for a finite-length solenoid like this, due to fringing field.



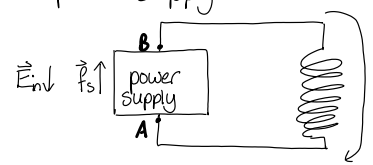
But we can get a decent approximation from the approximation that 9 Tesla is uniform within:

$$\Phi = \int \vec{B}\cdot d\vec{a} = B \cdot N\pi\left(\frac{D}{2}\right)^2 = (9\,T)(53{,}900)\pi\left(\frac{0.05\,m}{2}\right)^2 = 952\ T\cdot m^2$$

$$L = \Phi/I = (952\,T\cdot m^2)/(80\,A) = 12\ H$$

Now we have a 5V power supply

assume ideal power supply (internal $r=0$) $\Rightarrow |\vec{E}_{in}| = |\vec{f}_s|$



Ideal power supply $\Rightarrow$ no work done within power supply $\Rightarrow \vec{E}_{in} = -\vec{f}_s$

$$\mathcal{E} = \oint \vec{f}\cdot d\vec{l} = \oint(\vec{f}_s + \vec{E})\cdot d\vec{l} = \underbrace{\oint \vec{f}_s \cdot d\vec{l}}_{5\,V} + \underbrace{\oint \vec{E}\cdot d\vec{l}}_{-\frac{d\Phi}{dt} = -L\frac{dI}{dt}} = 0$$

$$\int_A^B \vec{f}_s \cdot d\vec{l} = V_B - V_A = 5\,V$$

$$\Rightarrow \frac{dI}{dt} = \frac{5\,V}{12\,H} = 0.42\ A/s$$

$\Rightarrow$ to ramp all the way up to 80A requires $\Delta t = (80\,A)/(0.42\,A/s) = 3$ min, 12 sec

Energy: The power supply had to do work against the back EMF. On a single charge going around the circuit once, the power supply does work per charge $\mathcal{E}_{battery} = 5V$ or we can think of this as work done against $\mathcal{E}_{magnet}$, so work is $-\mathcal{E}_{magnet} = -(-L\,dI/dt) = L\,dI/dt$.

Work per unit time = (work per charge)(charges per time)

$$\frac{dW}{dt} = \left(L\frac{dI}{dt}\right)(I)$$

$$W_{tot} = L\int_{t_0}^{t_{final}} I\frac{dI}{dt}\,dt = L\int_{I_0}^{I_{final}} I\,dI = \frac{1}{2}LI^2 \quad \text{(starting from } I_0 = 0\text{)}$$

Where is the energy actually stored? In the $\vec{B}$-field. (Unlike a power supply driving a resistor and converting energy to heat, this energy is stored in $\vec{B}$ and is recoverable.) We can derive that (see Griffiths 317-318):

$$W = \frac{1}{2}LI^2 \quad \rightarrow \quad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

Example: What happens if my superconducting magnet "quenches" i.e. suddenly goes normal, so there is suddenly an $R \approx 1\,k\Omega$.

Answer: a lot of energy is rapidly converted to heat!

stored $E = \frac{1}{2}(12\,H)(80\,A)^2 = 38.4\,kJ$

Latent heat of liquid helium = 21 J/g

Density = 0.125 g/mL

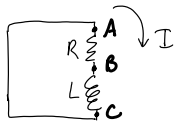
$$38.4\,kJ \Big/ \left(\frac{21\,J}{g} \cdot \frac{0.125\,g}{mL} \cdot \frac{1000\,mL}{1\,L}\right) = 14.6\,L \text{ boiled rapidly}$$

$$1\,L\ liq \cdot \frac{0.125\,kg}{1\,L} \cdot \frac{1000\,g}{1\,kg} \cdot \frac{1\,mol}{4\,g} \cdot \frac{22.4\,L\ gas}{1\,mol} = 700\,L\ gas$$

$\Rightarrow$ quench produces $(14.6)(700\,L) = 10^4\,L$ gas <u>very quickly</u>

How quickly?

Answer: the power supply is out of the circuit now (we disconnected it as soon as we were done ramping to 80A).



$$IR + \underbrace{L\frac{dI}{dt}}_{} = 0$$

Lenz' law: this is forward-EMF now: wants to keep the $\vec{B}$-field from changing

$$\Rightarrow \frac{dI}{I} = \frac{-R}{L}dt \Rightarrow \int_{I_0}^{I} \frac{dI'}{I_0} = \int_{t_0=0}^{t} -\frac{R}{L}dt'$$

$$\Rightarrow \ln\left(\frac{I}{I_0}\right) = \frac{-R}{L}t$$

$$\Rightarrow I = I_0 e^{-(R/L)t}$$

time constant $\tau = L/R = (12\,H)/(1\,k\Omega) = 12\,ms$

$\Rightarrow$ In 12 ms, I has gone from its starting value 80 A down to $\frac{1}{e}(80\,A) = 30\,A$

$\Rightarrow$ In 12 ms, the energy dumped was $\Delta E = \frac{80^2 - 30^2}{80^2}(38.4\,kJ) = 33\,kJ = 86\%$

$\Rightarrow$ We're still producing almost 10^4 L of gas in 12 ms!

Maxwell's Equations <u>SO FAR</u>

(i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss' law)

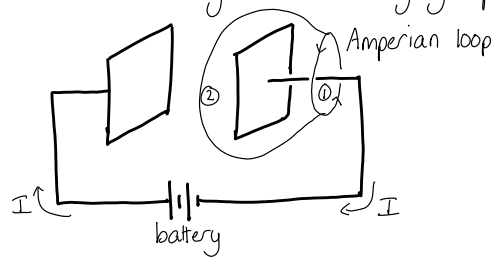
(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (Faraday's law)

(iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's law)

Are we done?

Consider the following: we're charging up a capacitor



Around the loop: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

But which surface should we use to get I_{enc}?

① obvious surface in the plane of the loop gives $I_{enc} = I$

② balloon surface passing between capacitor plates gives $I_{enc} = 0$ which is obviously wrong

But Stokes' thm says <u>mathematically</u> that it's ok to use either surface! So we must be missing some physics.

Look back at Maxwell (iv): $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$$\Rightarrow \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{\text{mathematically, divergence of a curl} = 0} = \mu_0 \underbrace{\vec{\nabla}\cdot\vec{J}}$$

but by the continuity equation

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}\cdot\vec{E}) = -\vec{\nabla}\cdot\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

Gauss' law

So what if we add $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ to $\vec{J}$ to correct (iv)?

$$\Rightarrow \text{new (iv)}\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Now we have fully consistent Maxwell's eqns, and we see that a changing $\vec{E}$ field produces a $\vec{B}$ field (like a changing $\vec{B}$ produces $\vec{E}$).

Next time: wrap up Maxwell's equations in matter

search for magnetic monopoles?